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GENERAL EQUILIBRIUM MODELS - NUMERICAL METHOD AND STABILITY

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ABSTRACT: Dynamic general equilibrium models are studied from a mathematical point of view. Study concentrates on conditions of optimality, stability and especially numerical solution algorithms. Models are also studied in respect of suitability for expanding the generational accounting method. In addition to Ramsey model two dynamic self-made models are presented, one of which uses the overlapping generations structure. Technological process is simulated with these models using various numerical methods. Also methods of analysing stability properties are presented and executed. At the end of the study several weaknesses of the general equilibrium analysis and suggestions how to overcome these weaknesses are discussed and a model structure suitable for expanding the generational accounting is presented.

KEY WORDS: Dynamic equilibrium models, overlapping generations, numerical methods, stability.

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TIIVISTELMÄ: Työssä tutkitaan dynaamisia yleisen tasapainon malleja matemaattisista lähtökodista eli painotetaan optimointia, stabiilisuutta ja erityisesti numeerisia ratkaisumenetelmiä. Työssä tutkitaan myös millainen yleinen tasapainomalli sopisi sukupolvitilinpitolaskelmiin. Työssä käytetään Ramseyn mallin lisäksi kahta tätä tutkimusta varten tehtyä yleisen tasapainon mallia, joista toisessa on mallinettu limittäiset sukupolvet. Malleilla simuloidaan teknistä kehitystä käyttäen erilaisia numeerisia menetelmiä. Malleissa käytetyt ulkoiset muuttujat ovat Kenc et al. estimoineet yleisesti Suomea varten. Työssä käydään läpi ja sovelletaan myös menetelmiä, joilla voi tutkia kyseisten mallien stabiilisuutta. Työn lopussa käydään läpi tasapainomallilähestymistavan puutteita ja tehdään ehdotuksia niiden poistamiseksi sekä ehdotetaan mallirakennetta, jota voitaisiin käyttää apuna sukupolvitilinpitolaskelmissa.

ASIASANAT: Dynaamiset tasapainomallit, limittäiset sukupolvet, numeeriset menetelmät, stabiilisuus.

Preface

Dynamic general equilibrium models have been a major method in economic theory since early 1960's. The problem was that only some simple models had analytical solutions and they were not feasible for policy simulations. At late 1980's advantages in computer and software technology enabled extensive use of computable dynamic general equilibrium models as a tool of policy simulations. Instead of analytical solutions the computable dynamic general equilibrium models are solved numerically using a specific iterative process.

The current economic recession has sparked a fear in the OECD countries during the past few years that low economic growth together with ageing baby-boomers might lead to a situation where the promises of public pension schemes cannot be fulfilled without enormous tax increases. This had led universities and government organisations of various countries to develop computable dynamic general equilibrium models, which use overlapping generations formulation to analyse this particular problem. In addition to public pension schemes various studies of tax reforms have utilised dynamic general equilibrium models, e.g., Auerbach-Kotlikoff's Dynamic Fiscal Policy. Also OECD uses overlapping generations method in a project, which analyses problems of ageing population.

Government Institute for Economic Research has developed dynamic general equilibrium models form 1992 onwards. This study by Antti Lappeteläinen mathematical aspects of the computable dynamic general equilibrium models are analysed. The study introduces several methods, which can be used to solve dynamic equations and methods for analyse stability properties of the equilibrium models as well.

The study is a part of a project to construct a large scale dynamic general equilibrium model on problems on Finnish pension schemes. The project is led by Prof. W. Perraudin, Birkbeck Collage and CEPR whose study will be published on spring 1995. He has also alongside with Senior researcher Reijo Vanne and Assistant Prof. Jukka Ruusunen helped this study with their comments. The study is also accepted as Antti Lappeteläinen's Masters thesis at System- and operations research laboratory at Helsinki University of Technology. The project is funded by the Finnish national fund for research and development, SITRA. On behalf of Government institute for Economic Research I want to thank every one involved on the project especially Antti Lappeteläinen.

Helsinki, on 21th April 1995

Seppo Leppänen

Esipuhe

Dynaamiset yleisen tasapainon mallit ovat kuuluneet olennaisena osana taloustieteen teoriaan 1960-luvun alusta. Ongelmana tuolloin oli, että analyyttisesti pystyttiin ratkaisemaan vain helppoja malleja, jotka eivät täytä simulaatiomalleille asetettuja vaatimuksia. Tietokone- ja ohjelmistotekniikan kehitys 1980- luvun lopussa mahdollisti laajojen dynaamisten tasapainomallien käytön politiikkasimulaatioissa. Numeeriset mallit poikkeavat analyyttisistä malleista siinä että analyyttisen ratkaisun sijasta tehtävä ratkaistaan numeerisesti joissain pisteissä jollakin numeerisella algoritmilla.

Talouden nykyinen taantuma läntisissä teollisuusmaissa herätti pelkoja, että hidas taloudellinen kasvu yhdessä ikääntyvien suurten ikäluokkien kanssa vie pohjan julkisilta eläkelupauksilta ilman suuria veronkorotuksia. Tämä on saanut monet korkeakoulut ja tutkimuskeskukset kehittämään limittäisten sukupolvien dynaamisia tasapainomalleja. Eläketutkimuksen lisäksi monet verouudistuksia käsittelevät tutkimukset käyttävät dynaamisia tasapainomalleja, esimerkiksi Auerbach-Kotlikoffin "Dynamic Fiscal Policy". Myös OECD käyttää tämän tutkimussuunnan keskeisiä työvälineitä ikääntyvän väestön taloudellisia vaikutuksia käsittelevässä hankkeessa.

Valtion taloudellinen tutkimuskeskus on vuodesta 1992 lähtien panostanut dynaamisten tasapainomallien kehittämiseen. Tässä Antti Lappeteläisen laatimassa tutkimuksessa tarkastellaan numeerisia dynaamisia tasapainomalleja matemaattisista lähtökohdista. Tutkimuksessa arvioidaan eri algoritmien sopivuutta numeeristen dynaamisten tasapainomallien ratkaisemiseen sekä mallien stabiilisuusominaisuuksia.

Tutkimus on osa projektia, jossa tarkastellaan Suomen eläkejärjestelmän kestävyyttä dynaamisen tasapainomallin avulla. Prof. W. Perraudin, joka toimii professorina Birkbeck College Lontoossa, on laatimassa tästä aiheesta keväällä 1995 ilmestyvän tutkimuksen. Hän on myös ohjannut osaltaan tämän tutkimuksen tekoa. Antti

Lappeteläistä ovat myös auttaneet kommenteillaan erikoistutkija Reijo Vanne sekä apulaisprofessori Jukka Ruusunen. Tutkimus on hyväksytty Teknillisen korkeakoulun tietotekniikan osaston systeemi- ja operaatiotutkimuksen laboratorion diplomityöksi. Hankkeeseen on saatu rahoitusta SITRAlta. Valtion taloudellisen tutkimuskeskuksen puolesta esitän kaikille hanketta eteenpäin vieneille ja erityisesti Antti Lappeteläiselle parhaat kiitokset.

Helsingissä 21.4. 1995

Seppo Leppänen

Contents

1 INTRODUCTION

In this study we analyse dynamic general equilibrium models from a mathematical rather than an economic point of view. In dynamic general equilibrium models supply and demand decisions by economic agents are functions which maximise agents' intertemporal utility or profits, usually under the assumption of perfect foresight[1]. In addition the balance between the supply and demand of each good must be fulfilled. The dynamic structure allows us to consider economic issues like intergenerational redistribution of utility, which we cannot analyse using static models. The mathematical point of view means here that we are interested in mathematical problems such as numerical solutions necessary conditions for optimality and stability. Normal economic considerations related to dynamic general equilibrium models such as the effects of the choice of tax base on agents [Auerbach & Kotlikoff (1987)] [Perraudin & Pujol (1991)] are not focused in this study.

The main purpose of this study is to analyse dynamic general equilibrium models that could be used for generational accounting[2] and would preferably have the following properties. The model should tell how the wage rate, level of savings and labour demand of different generations react to economic growth or government policies. The model should be reasonably easy to solve numerically and there should be a simple method for analysing its stability properties. We have divided our main purpose into three themes: numerical methods for solving dynamic general equilibrium models, the constructing simple dynamic equilibrium models and obtaining the conditions of optimal behaviour by economic agents in these models under the assumption of perfect foresight and reviewing and applying some of the theories and methods for analysing the stability properties.

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1 Implying that agents have perfect knowledge of all future prices.

2 Generational accounting is defined in the Appendix 2.

This study is organised as follows. In chapter 2 we analyse different numerical methods for solving dynamical general equilibrium models. Apart from the simplest equilibrium models no analytical solutions generally exist. Hence numerical methods must be applied in almost all policy simulations using the dynamic general equilibrium models. For numerical methods we introduce the Ramsey model, also known as the optimal growth model [Azariadis (1993)] and [Sargent (1987)], which is a simple model with a long-lived agents structure but very useful for testing the various numerical methods. We apply six different numerical methods to the Ramsey model and analyse the advantages and disadvantages of these numerical methods with respect to the Ramsey model and consider what kind of difficulties might ensue if these methods are applied to more complicated equilibrium models. The numerical methods are basic, multiple and bounded shooting methods, normal and two modified Wilcoxen methods. One of the modified Wilcoxen methods uses the Jacobi iteration algorithm and the other the Gauss-Seidel iteration algorithm. [Lipton et. al. (1982)], [Keuschnigg (1991)], [Spender (1985)], [Bertsekas & Tsitsiklis (1989)].

In chapter 3 we construct dynamic general equilibrium models. We explain the differences between long-lived agents models and overlapping generations models, derive the equations for optimal behaviour using the Kuhn-Tucker conditions [Bazaara & Shetty (1979)] and recursive dynamic programming [Sargent (1987)]. We discuss how the optimality equations of the long-lived agents model and the overlapping generations model must be formulated in order to apply numerical methods [Cazes et al.(1992)] and [Kendrick (1981)]. All the models in this study have discrete dynamic structures.

In chapter 3 we construct two dynamic models: one long-lived agents model and one overlapping generations model. The models include intertemporal utility maximisation by households, in respect of consumption and leisure abilities, and a competitive firm that maximises its profits by paying for its inputs, labour and capital according to their marginal productivity. The main idea and the economic background for these models are taken from Auerbach & Kotlikoff (1987).

We use these models to simulate a technological innovation which increases productivity. We analyse its effects on wages, interest rates and savings and consumption by households by solving the models by rearranging the equations in them so that Jacobi iteration can be applied.

In chapter 4 we review mathematical methods of analysing whether a stationary solution of a non-linear difference equation is a saddle point. We show that there is a unique transition path to the fixed point of the system of difference equations [Laitner (1984) and (1990)]. The fixed point is called the final steady state when it is associated with dynamic general equilibrium models. We also review the theory of the bifurcations of equilibrium [Azariadis (1993)] and comparative statistics analysis around the final steady state. At the end of the chapter we discuss sensitivity analysis, which is a practical solution for assessing how changes in exogenous variables affect the position of the final steady state and the transition path [Auerbach & Kotlikoff (1987)], [Perraudin & Pujol (1991)]. The main mathematical tool of this chapter is the implicit function theorem [Rudin (1989)].

We apply the uniqueness and sensitivity methods to the final steady state of our overlapping generations model to study how these methods can be applied numerically.

In the last chapter we draw conclusions. We discuss what kind of dynamic general equilibrium model would be best suited to generational accounting for it to forecast savings and wage rates and labour supply trends. At the end of the chapter we discuss the weaknesses of the dynamic general equilibrium models as tools of analysing policy simulations.

The essential part of constructing the general dynamic equilibrium models is the parameter estimation. However this study does not discuss parameter estimation[3]. We have decided to omit statistics out of this study, because we want to concentrate on numerical methods optimisation and stability. The collection of time-series data alone for parameter estimation would also have been too time consuming.

The parameter values for the Ramsey model are chosen randomly and the parameter values for the long-lived agents model and for the overlapping generations model are mainly those estimated for Finland by Kenc et al. (1994). They use the statistical method known as the general method of moments to micro- and macro economic time-series data of Finland. Even though some of the exogenous parameters are estimated for a particular country, economic conclusions cannot be on the basis of these dynamic general equilibrium models. In this study we concentrate on the mathematical side of the dynamic general equilibrium models.

All the programs for the numerical solutions, e.g., the shooting methods, are our own Excel macros. We found the cell structure of Excel very convenient for discrete calculations, especially for programming the bounded and the multiple shooting method. The ease of using the graphics was another advantage of Excel. On the other hand the execution of Excel commands related to matrix operations was not so simple. It might possible have been easier with some another software package, e.g., Gauss.

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[3] "It is here that economics begins and computer science ends." [Auerbach et al. (1987), pp.54]

2 NUMERICAL METHODS FOR THE DYNAMIC GENERAL EQUILIBRIUM MODELS

In this chapter we present a simple perfect foresight dynamic general equilibrium model with an analytical solution. The model is known in economics literature as the optimal growth model or the Ramsey model. Secondly we formulate the model so that numerical methods can be applied. Thirdly we solve the model using three types of shooting methods: basic, multiple and bounded and three types of Fair-Taylor algorithms. During and after solving the model with different algorithms we discuss the advantages and disadvantages of each of the algorithms. We also discuss how the performance of the different algorithms varies with problems which have more dimensions than the optimal growth model.

2.1 The optimal growth model

The optimal growth model we use was pioneered by Ramsey (1928) and later developed by Cass (1965) and Koopmans (1965). The model is quite simple, but particularly useful for test numerical procedures, while a non-trivial analytical solution exists [Sargent (1987)]. Also for those with a mathematical rather than an economics background the optimal growth model describes how economists map the economy using difference equations and analyse it using discrete dynamic optimisation techniques.

The model is a discrete mapping of a stationary economy with a fixed labour supply and capital adapted freely without any transition costs. Only one consumption good[4] is produced and it is described by the Cobb-Douglas production function [Chiang (1984), pp. 414]. The economy consists of a single infinitely lived individual with perfect foresight. We set "the clocks" so that he is born at t=0. He will not be paid

[4] It is useful to think of the production good as wheat, which can be either eaten (consumed) or stored for farming next year (invested). In this model we refer K_{t+1} as investments in the period t and prodution capital in period t+1.

for his labour, but as the owner of the production capital he will own all the goods produced. At the beginning of each period the individual is faced with deciding the optimal consumption/investment ratio to maximise his intertemporal utility function:

[2.1] $$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

under the constraint that investments (K_{t+1}) and consumption (c_t) cannot exceed output $f(K_t)$, which is a function of capital invested for production in the preceding period.

[2.2] $$K_{t+1} + c_t = f(K_t)$$

The variable β is a discount factor, t is a time period index, c is consumption, k capital. The utility and production functions u(t) and f(k) have the following properties $u'(0) = f'(0) = \infty$, $f',u' > 0$, $f'',u'' < 0$, $u''(\infty)=0$. The equals sign in [2.2] instead of the less than sign assumes that no production is wasted. The single infinitely lived individual's maximising problem can also be considered dynastic behaviour, where the generation currently alive not only considers its own well-being, but also that of its descendants. However, the discounting term b in the utility function, indicates that the present generation alive is more concerned with itself than its offspring.

For our study let $u(c_t)=\ln c_t$ and $f(k_t) = AK_t$, where A is a scaling constant, and $0<\alpha<1$ is a factor implying decreasing returns to scale. In the farmer's example the farmer is not able to sow and cultivate large amounts of wheat as effectively as smaller amounts. Mathematically this means that together with the linearity of [2.2] with respect to c_t, the Kuhn-Tucker sufficient conditions for the maximum are satisfied. Under the assumption of perfect foresight the individual is able to maximise his intertemporal utility the "day" he is born. To maximise [2.1] under constraint [2.2] we write a function

[2.3] $$L = u(c_t) + .. + u(c_{t+n}) + .. + \lambda_0(AK_t - c_t - K_{t+1}) + ..,$$

which is analogous to the Hamiltonian function in continuous time [Kendrick (1981)], [Sargent (1987)]. The l's are Lagrange multipliers associated with budget constraints for each period. We define c_t as a control variable and K_t as a state variable. We can write the first order conditions:

[2.4 a,b]
$$\frac{\partial L}{\partial c_t} = \beta^t \frac{1}{c_t} - \lambda_t = 0$$
$$\frac{\partial L}{\partial k_t} = \lambda_t \alpha A K_t^{\alpha-1} - \lambda_{t-1} = 0$$

Because our end point conditions are: 1. end time fixed (infinity), 2. end state free, the boundary conditions, which are also called transversality conditions, reduce to[5] [Kamien & Schwartz(1991)].

[2.4 c] $$\lim_{T\to\infty} \lambda_T \to 0.$$

To form a relation $c_{t+1} = G(c_t)$ we divide [2.4.b] by itself lagged one variable and replace lt-1 by 2.4.b. Together with the budget constraint, it leads to the following discrete dynamic system:

[2.5 a,b]
$$K_{t+1} = AK_t^{\alpha} - c_t$$
$$c_{t+1} = A\alpha\beta(AK_t^{\alpha} - c_t)^{\alpha-1} c_t ,$$

where k_0 is given and $c_{t+1} = c_t$ when $t > T$.

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5 Provided there is a discounting term $\beta < 1$. According to [Barro & Sala-i-Martin (1994)] an infinite end point can be treated similarly to a finite endpoint. However there is no proof of this.

Equation [2.5.b] is considered as a discrete time version of the solution of the Euler-Lagrange equation for the necessary condition for the extremum in continuous time [Kendrick (1981)], [Kirk (1970)]. Therefore it is often referred in economics literature to the Euler equation.

This is the standard method for solving discrete dynamic general equilibrium models. However, the optimal growth model is simple enough to be solved analytically [Sargent (1987)]. Instead of the optimal control approach we used here, Sargent uses the dynamic programming developed by Richard Bellman. The solution method is based on defining K_{t+1} as a control variable and K_t as a state variable. Next Sargent (1987) replaces c_t in [2.1] by [2.2] and forms the Bellman equation associated with this problem. The Bellman equation satisfies Bellman's principle of optimality, where $v(K_t)$ is the value function:

$$v(K_t) = \max_{K_{T+1}} \{\ln(AK_t^{\alpha} - K_{t+1}) + \beta v(K_{t+1})\}.$$

Sargent (1987) solves the Bellman equation by making a guess $v(K_t) = E + F \ln K_t$ for the value function $v(K_t)$ and verifying it. The analytical solution to capital accumulation in this version of the optimal growth model is:

[2.6] $$K_{t+1} = A\alpha\beta K_t^{\alpha}$$

The questions studied by the optimal growth model are, for example, related to growth theory [Barro & Sala-i-Martin (1994)] which analyses changes in production and how fast these take place, if technological change takes place. Consider the parameter A in the production function ($f(K_t) = AK_t$) to be constant for many periods, say $A=2.5$, $\beta=.8$ and $\alpha=0.5$. In that case optimal policy would be to remain in a steady state and produce 2.5 units and consume 1.5 units in each period. Then suddenly a technological innovation takes place and A becomes four times as large. By solving the model we find out that consumption will rise from 1.5 to 16, but it takes 18 periods (using three decimals) to achive a new steady state. The optimal growth model is only a theoretical model and the questions are also quite theoretical. In chapter 3 we deal with more complicated models which can be used to predict e.g. how the similar technological innovation effects the consumption patterns of the individuals in that economy and the demand for labour.

We showed earlier that the discount factor $\beta < 1$ is correct, on the ground that the current generation prefers itself to the future generations. Also the mathematics of the optimal growth model "forces man to do so". Consider a more selfishness case where $\beta = 1$. The equation [2.6] converges to a finite positive number k* (if $0 < \alpha < 1$). Then equation [2.5.a] implies that there is a finite positive number c* for steady-state consumption. According to equation [2.4.a]:

$$\lim_{t \to \infty} \lambda_t \to \frac{1}{c^*} > 0.$$

Thus the transversality condition [2.4.c] is violated. Thus according to the optimal conditions of the optimal growth model, the current generation is more concerned with itself than with its offspring[6].

2.2 Shooting methods

Shooting methods are well known procedures both in the engineering and economics fields. They can be applied to both differential and difference equation systems. The basic idea of all shooting methods is to guess a set of values, including the values of the first period. Then the system is "shot", or integrated forward in order to verify whether the guessed values satisfy the terminal conditions, which are the final steady-state requirements in the optimal growth model. Thus we want to end up with constant and positive levels of consumption and savings after a certain period. If

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[6] Barro & Sala-i-Martin (1994) discuss the transversality conditions of the infinite horizon Ramsey model in continuous time. According to them the sufficient transversality condition in the case of $\beta = 1$ is $\lim(t \to \infty)$ H(t)-> 0 , where H is the Hamiltonian of the problem.

these are not satisfied, we use iteration methods such as Newton's method[7] to improve the guesses.

As an example let us consider a system of difference equations,

[2.7] $x_{t+1}=F_t(x_t,z_t)$,

where x_t is an unknown n*1 vector and z_t is a known k*1 vector. In the case of our optimal growth model the values of z_t are the parameters A, α and β, which in this case are constants. Let k_t and c_t be subvectors of x_t with dimensions of m*1 and (n-m)*1 respectively. As the boundary conditions we have:

[2.8 a,b]
$$k_0=\bar{k}, \text{ and}$$
$$G(x_{T-1}, x_T)=0$$

The second boundary condition is an approximation of $\lim_{T\to\infty} x_T - x_{T-1} = 0$ in infinite horizon models.

In the basic shooting method only the initial values (c_0) are guessed. This requires the iteration of n-m scalars. In systems with non-integer power terms, such as dynamic general equilibrium models, the execution of the basic shooting method algorithms is usually terminated by the error message "cannot raise negative number to non-integer power", or the series diverge to infinity. In optimal growth this is due to the fact that the steady state is a saddle point. This will take place even with accurate initial values and when using a relaxation parameter to generate new values in the iteration.

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7 To find the solution to the system f(x)=0, the Newton method is $x_{t+1}=x_t-(H(x_t))^{-1}\tilde{N}f(x_t)$, where H is the Hessian matrix of f.

To avoid these problems, multiple shooting divides the solution path into subintervals of s time periods in length, which are solved such that the values of the preceding subinterval do not effect the calculation. The start of succeeding subinterval is used as a terminal condition for the preceding interval. In addition to iterating s*(n-m) control variables (c) we also have to iterate (s-1)*m state variables (k). When the subintervals are small enough this method proves to be a very powerful tool but is at the expense of increasing the dimensions of the calculations in the iteration process. In large-scale economic simulations the use of the multiple shooting method leads to several transformations of huge matrices [Lipton et al.(1982)]. This can be avoided by using the quasi-Newton method [Bazaara & Shetty (1979)] where matrix transformation takes place only at the beginning of an iteration process and the same Hessian matrix is used at later iteration rounds. But this could lead to problems if the initial values are poorly chosen.

A good compromise between the basic and multiple shooting methods is the bounded shooting method [Spencer (1985)]. In the bounded shooting method we only iterate (n-m) scalars, as in the basic shooting method. The integration or "shot" is then executed until period t, where the first auxiliary boundary condition

[2.9] $$g_t^{lower} < G_t(c_{t-1}, c_t) < g_t^{upper}$$

is breached. The choice of the upper and lower boundaries of auxiliary boundary conditions is largely a matter of experimentation. In the optimal growth model the difference between ct and ct+1 narrows sharply after a few periods, due to the decreasing returns in the production function [Varian(1992)]. This would suggest that the boundaries of the auxiliary boundary conditions should be in the form of narrowing "tubes" as shown in figure 2.1. According to Spencer (1985), the bounded shooting method also works well in large-scale economic models. It has been applied in models used by the Bank of England.

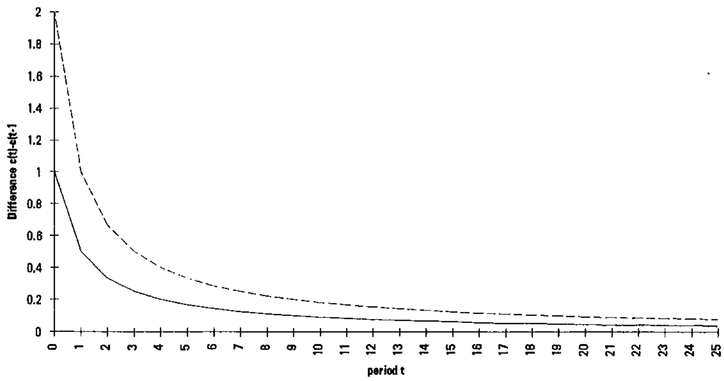

Figure 2.1: Illustration of the shape of "a tube" for the auxiliary boundary condition of optimal growth models.

2.2.1 Algorithm for the Basic Shooting Method

Applying the basic shooting method to a problem such as the optimal growth model starts with an initial guess for c_0, integrates [2.7] forward with the given [2.8.a] to derive an implied violation for [2.8.b]. In our applications we require a steady-state at period T. Therefore we let the boundary condition [2.8.b] be

$$G(c_0) = c_T(c_0) - c_{T-1}(c_0) ,$$

where $c_T(c_0)$ maps $c_0 \rightarrow c_T$ with the given k_0, and defining $H(c_0)$ as a Jacobian of $G(c_0)$ at point c_0. We assume there is a non-trivial solution at $c_0^* > 0$. We can write a linear relation between the initial guess c_0 and the solution c_0^* by using the Taylor approximation in the neighbourhood of c_0^*.

$$c_T(c_0^*) \approx c_T(c_0) + \frac{\partial c_T(c_0)}{\partial c_0}\Big|_{c_0}(c_0^* - c_0)$$

[2.10] $$c_0^* \approx c_0 - \left[\frac{\partial c_T(c_0)}{\partial \hat{c}_0}\Big|_{c_0}\right]^{-1}(c_T(c_0) - c_T(c_0^*)).$$

The problem is that we do not know the end point value $c_T(c*0\ )$. So we must approximate error $c_T\ (c_0\ )$- $c_T\ (c_0*)$ by $c_T(c_0)$- $c_{T-1}(c_0)$.

[2.11] $$c_0^* \approx c_0 - \left[\frac{\partial c_T(c_0)}{\partial \hat{c}_0} |_{c_0} \right]^{-1} (c_T(c_0) - c_{T-1}(c_0))$$

If we use the notation defined above we can rewrite [2.11]:

[2.12] $$c_0^* \approx c_0 - [H]^{-1} G_T$$

If the problem is linear, the Taylor approximation is exact. If not, we can apply an iterative process to achieve convergence:

[2.13] $$c_0^{n+1} \approx c_0^n - [H]^{-1} G_T,$$

where H is evaluated at C_0^n . To improve the probability of finding the solution, we highly recommend using an relaxation parameter even in simple optimal growth models. With a relaxation parameter equation [2.13] becomes:

[2.14] $$c_0^{n+1} \approx c_0^n - r(i)[H]^{-1} G_T$$

where the relaxation parameter $0 < r(i) < 1$ is a function of the iteration round i.

2.2.2 Algorithm for the Multiple Shooting Method

To apply the multiple shooting method we divide the interval [0,T] to subintervals $[0,T_1],[T_1,T_2]\ldots[T_{s-1},T_s]$, with T_s=T. The problem defined in [2.7] can be characterised as:

[2.15]

$$X_{t_1} = H_1(K_0, \hat{c}_0)$$

$$X_{t_2} = H_2(\hat{K}_1, \hat{c}_1)$$

$$\dots$$

$$X_s = H_s(\hat{K}_{s-1}, \hat{c}_{s-1})$$

The variables with hats are unknown variables of the problem. Thus $\hat{K}_{t_1}, \hat{c}_{t_1}$ are starting vectors of interval $[T_1,T_2]$ and $H_2(\hat{K}_1, \hat{c}_1)$ is the value of vector X at the end of interval $[T_1,T_2]$. The dimensions of H_t differ. H_1 maps $R^{(n-m)*1} \rightarrow R^n$ and for $i=2,\dots,s$ H_i maps $R^n \rightarrow R^n$. After introducing matrix notation:

$$\hat{\mathbf{X}} = \left[\hat{c}_0, \hat{X}_1, \hat{X}_2, \dots \hat{X}_s\right] \text{ and}$$

$$\mathbf{X} = [\hat{c}_0, X_1, X_2, \dots X_s], \text{ where } \hat{X}_i = \left[\hat{K}_i, \hat{c}_i\right],$$

the system [2.15] can be rewritten as :

[2.15 b] $$X = \tilde{H}(\hat{X})$$

with $\tilde{H}$: $R^{(n+(s-1)+(n-m))} \rightarrow R^{(n+(s-1)+(n-m))}$. For the matrix notation it is also useful to define a matrix M as:

$$\mathbf{M} = \left[X_1 - \hat{X}_1, X_2 - \hat{X}_2, \dots G(X_{t-1}, X_t\right. .$$

Since X is a function of $\hat{x}$, M is an implicit function of $\hat{x}$. If we can locate $\hat{x}^*$ so that

[2.16] $$M(\hat{X}^*) = 0^8$$

then $\hat{c}_0$ satisfies equations [2.15] and [2.8 a,b]. To locate $\hat{X}^*$ we use Newton's search as in the basic shooting method.

8 According to Brouwer's theorem [Bertsekas & Tsitsiklis (1989)] the equation [2.16] has a solution.

$$[2.17] \qquad \hat{\mathbf{X}}^{i+1} \approx \hat{\mathbf{X}}^{i} - \left(\frac{\partial \mathbf{M}}{\partial \hat{\mathbf{X}}}\right)^{-1}_{\hat{\mathbf{X}}^{i}} \mathbf{M}(\hat{\mathbf{X}}^{i})$$

Each iteration of [2.17] requires an inversion of a [n(s-1)+(m-n)] matrix. For large and complicated systems many subintervals are required. This may become very costly computationally. Fortunately $\left(\frac{\partial \mathbf{M}}{\partial \hat{\mathbf{X}}}\right)$ can be written:

$$[2.18] \qquad \frac{\partial \mathbf{M}}{\partial \hat{\mathbf{X}}} = \begin{bmatrix} \frac{\partial H_1}{\partial \hat{c}_0} & -I & 0 & \cdots & \cdot & 0 \\ 0 & \frac{\partial H_2}{\partial \hat{X}_1} & -I & 0 & 0 & 0 \\ \cdots & & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & & \frac{\partial H_{s-1}}{\partial \hat{X}_{s-2}} & -I \\ 0 & \cdots & \cdots & & & \frac{\partial H_s}{\partial \hat{X}_{s-1}} \end{bmatrix}$$

where I is the n´n identity matrix. $\frac{\partial H_1}{\partial \hat{c}_0}$ has the dimensions (m-n)*n, $\frac{\partial H_s}{\partial \hat{X}_{s-1}}$ has the dimension n*(m-n) and the rest of the blocks have dimensions of n*n. This form affords substantial computational savings in the inversion procedure. From now on we refer to equation [2.18] as the H-matrix.

2.2.3 Algorithm for the Bounded Shooting Method

The bounded shooting method differs from the basic shooting method only in that [2.7] is integrated or "shot" forward with the initial guess c_0 only until the auxiliary boundary condition [2.9] is breached and not necessary till the T^{th} period. The implied violation of [2.8] is derived at the point where the auxiliary boundary condition [2.9] is breached. This leads to an iterative process similar to that in the basic shooting method.

[2.19] $$c_0^{n+1} \approx c_0^n - [H]^{-1} G_\tau$$

where τ is equal to T if the auxiliary boundary condition [2.9] is not breached. If the auxiliary boundary condition is breached, τ is the period in which the auxiliary boundary condition is first breached and H is a Jacobian:

[2.20] $$H = \frac{\partial c_\tau(c_0)}{\partial c_0} \vdots_{c_0}$$

and

$$G = G(x_{\tau-1}, x_\tau).$$

2.2.4 Performance of the Shooting Method Algorithms

We use the optimal growth model described in section 2.1. The problem is to solve equations [2.6] and [2.7] so that with the given k_0 the steady-state requirement [3.0] is satisfied. For numerical purposes we set T=25, A=10, α=0.5, β=0.8 and k_0=1. In the multiple shooting method T was divided to eight subintervals. Hence every subinterval consists of four time periods. We show the analytical solution of the optimal growth model with these parameters in figure 2.2.

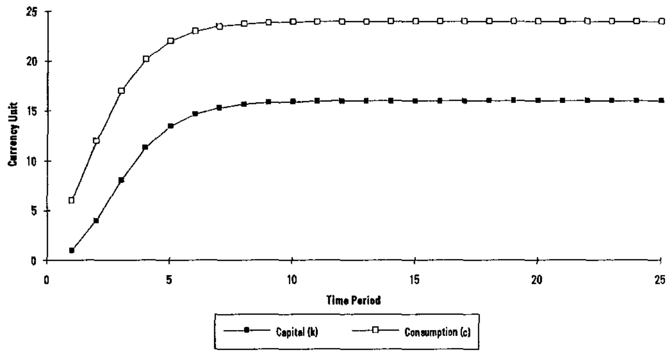

Figure 2.2: Analytical solution for the transition paths of the optimal growth model with parameters as defined above.

The saddle point structure combined with the 15-digit numerical calculation cumulates error disabling the use of the basic shooting method when the transition period is long, even when the initial guess is almost correct. In addition using the basic shooting method without the relaxation parameter almost certainly leads to numerical overflow because if initial consumption is too low in the first iteration this leads to overconsumption in the first period in the second iteration round, eventually leading to negative capital in some period in the second iteration round. With our parameters and the initial condition, six units of consumption in the first period is the solution to the optimal growth model. And initial guess less than 5.99999 units of consumption leads to negative capital and numerical overflow in the second iteration round. So the basic shooting method without the relaxation parameter has hardly any practical value. The bounded shooting method without the relaxation parameter converges to the fixed point. But the convergence is slow, because the shooting integration must be interrupted because of the numerical overflows.

With the relaxation parameter, the basic shooting method works significantly better and it is also faster as long as the relaxation parameter is chosen so that there are no overflows and if it is also reasonably large. Figures 2.3 and 2.4 illustrate the roles of the relaxation parameter and the auxiliary boundary condition.

In figure 2.3 the value of the initial guess is on the x-axis and the outcome of one iteration round using the basic shooting method is on the y-axis. The normal curve in the figure indicates the outcomes without the relaxation parameter. As the normal line shows almost all initial guesses over 4 lead to overflow after one iteration because the outcome is over 6, which the basic shooting method cannot be deal with. The reason is that the mappings of equations [2.13] and [2.14] with c_0 over 6.0001 are not real numbers. Initial guesses below 4 also lead to overflow after two or more initial rounds, e.g. the outcome of 3 is 4.96 and the outcome of 4.96 is 6.27. The purpose of the efficient constant relaxation parameter is to find the best linear combination of a 45-line (y=x) and the normal line so that the combination (relaxation line) is as close to the normal line as possible but does not have values over 6. The relaxation line in figure 2.3 has a constant relaxation parameter of 0.5, which is a good approximation of the best possible constant relaxation parameter. The fact that the normal line and the 45-line cross at two points indicates that the fixed point equation [2.13] has two solutions, $c_0=0$ or $c_0=6$, and the latter is stable.

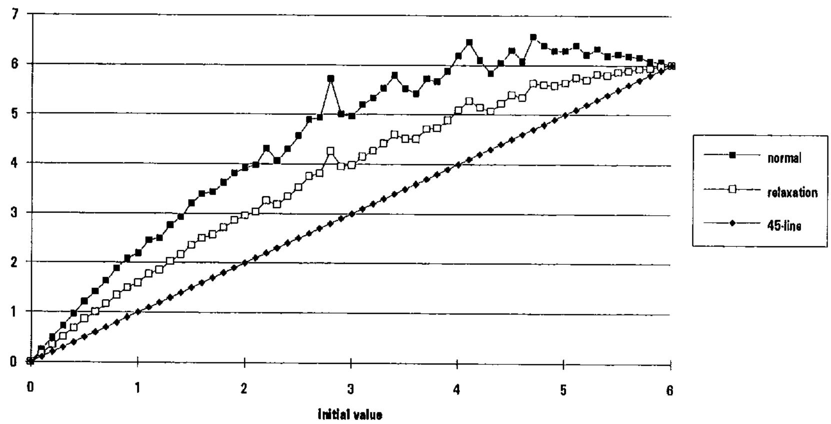

Figure 2.3: Mapping of the fixed point equations [2.13] and [2.14]. Basic shooting method.

Figure 2.4 shows the outcomes of the bounded shooting method with the auxiliary boundary condition to stop integration at period t, if c_t-c_{t-1} < -10 or if there is overflow at period t+1. The bounded shooting method does not terminate the iteration process if the input value implies overflows from period 2 onwards, so outcomes over 6 are not a problem. Figure 2.4 also shows that there is only one positive solution to the optimal growth problem.

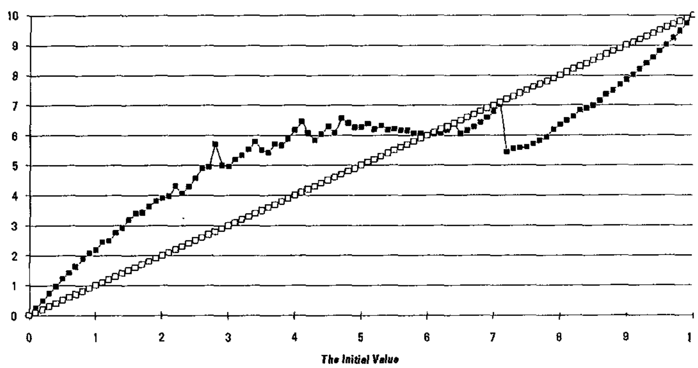

Figure 2.4: Mapping of the fixed point equation [2.19] with the auxiliary boundary condition -10 < $c_t(c_0)$-$c_{t-1}(c_0)$ < 10. Bounded shooting method.

In table 2.1 we show how different constant relaxation parameters affect the numbers in the iteration rounds needed for convergence in the basic shooting method. Note that the non-monotonic form of the normal line is the reason that fewer iteration rounds are needed for convergence from an initial value of 2.8 than from an initial value of 2.9.

The iteration results for the bounded shooting method are shown in table 2.2. As an upper boundary for the auxiliary boundary condition we stopped the "shot" at t if c_t-c_{t-1} > 10 or if there was an overflow at period t+1. As a lower boundary we tried three different auxiliary boundary conditions: stopping the "shot" at t if c_t-c_{t-1} < -10, c_t-c_{t-1} < -1 or c_t-c_{t-1} < -0.1.

For both the basic and the bounded shooting methods we used $|c_T(c_0)/c_{T-1}(c_0)-1| <$ 1*10e-6 as the criterion of convergence.

Table 2.1: Required iteration rounds for convergence of basic shooting method with different initial values and different relaxation parameters. Criterion for convergence $|c_T(c_0)/c_{T-1}(c_0)-1| <$ 1e-6.

Initial value	Relaxation parameter		
	1	1	1
8	overflow	overflow	overflow
7	overflow	overflow	overflow
4	51	38	overflow
3	53	overflow	overflow
3	52	39	overflow
0	64	48	overflow

Table 2.2: Required iteration rounds for convergence of bounded shooting method with different initial values and different upper bounds for the auxiliary boundary condition. Criterion for convergence $|c_T(c_0)/c_{T-1}(c_0)-1| <$ 1e-6.

Initial value	Lower limit of the auxiliary boundary condition		
	-10	-1	0
8	66	66	66
7	58	58	58
4	64	36	54
3	72	37	58
3	62	37	59
0	67	44	73

Comparing the bounded shooting and the basic shooting method in three major respects, namely programming effort, speed of convergence and tolerance of error in initial values, prefers the bounded shooting method to the basic shooting method in all the respects. If we consider problems of greater dimensions, the weight given to tolerance of different initial values must be superior to the weight given to programming effort. However, we must emphasise, that different relaxation parameters were given a *priori*, they were not solutions of any separate optimisation problem. That would improve the basic shooting method in speed of convergence but at the cost of more programming effort.

Programming the bounded shooting method algorithm is almost identical to programming the basic shooting method algorithm. Only a single extra loop is required, namely to check, whether the auxiliary boundary condition has been violated. Table 2.2 indicates that widening the tolerance of the auxiliary boundary condition will not terminate iteration. This would suggest that it is not necessary to devote much effort to the auxiliary boundary conditions. In the economic models functional forms seem to behave well except that negative values for state and control variables lead to numerical overflows.

With the second criterion, the speed of convergence, it is not clear which method to prefer. Since one iteration round takes longer with the bounded shooting method and because we chose the relaxation parameters and the auxiliary boundary conditions randomly. But intuitively we assume that the bounded shooting method converges faster, because the auxiliary boundary conditions keep the shots in the tube.

The tolerance range for initial value errors seems to be much better with the bounded shooting method algorithm even without the relaxation parameter than with the basic shooting method algorithm. This is why the bounded shooting method is generally referred to the basic shooting method. According to Bazaara & Shetty

(1979) Newton's method is guaranteed to converge regardless of the starting point if the Hessian matrix is symmetric positive definite and relaxation parameter chosen by auxiliary minimisation problem. In economic applications with non-integer powers, however, the Hessian matrix does not exist with all initial values, therefore even the best possible relaxation parameter function cannot be superior to the bounded shooting method.

The multiple shooting method differs from the other shooting methods, in that in addition to the initial values (c_0) we have guess and iterate values for all elements of the x-vector for the first time period of each subinterval, with the exception of the first subvector, where the k_0-vector is known. In practice this leads to sharp increase in the dimensions of the H-matrix. Inversion of the H-matrix gets computationally costly when the dimensions increase. Another factor slowing convergence is that in addition to [2.8.b], $x_s - \hat{x}_s = 0$ must hold for every s=[1,S-1]. The relaxation parameter function was kept the same for all iterations a(i)=025*i^2, where i is the number of the iteration round.

Table 2.3: Required iteration rounds for convergence of the multiple shooting method with different initial values and different frequency of inversion of the H-matrix. Criterion for convergence $|(c0)\text{-}cT\text{-}1\ (c0)|+\sum_{s=1}^{S-1} |\hat{c}_s - c_s| <1e\text{-}6$

	Frequency of calculation of inverse H-matrix		
Error in all initial values (%)	Every iteration round	Every second iteration round	Every fourth iteration round
10%	8	9	10
20%	8	10	12
30%	10	11	15
40%	10	13	overflow
50%	10	overflow	overflow
75%	overflow	overflow	overflow

Comparing the bounded and the multiple shooting methods is not as straightforward as comparing the bounded and the basic shooting methods. There are several reasons for that: different number of initial variables, different convergence criteria, and most of all the fact that each iteration round in the multiple shooting method requires the inversion of much larger matrices than in the bounded shooting method.

In table 2.3 we have analysed what happens if the H-matrix is not inverted in every iteration round and instead we use the previous iteration round values of the H-matrix. As seen from table 2.3, it is not necessary to obtain new values for the inverse H-matrix in every iteration round. At least this is true in straightforward economic problems with low relaxation parameter value at the beginning of the iteration process. But even if we use the same H-matrix for four consecutive iteration rounds, we still could not match the bounded shooting method in terms of time.

There are alternative methods [Bazaara & Shetty (1979)] for totally avoiding transformation of the H-matrix known as quasi-Newton methods. In these methods the H-matrix is approximated by various iterative processes. The purpose of the approximating the H-matrix is that $\lim_{i\to\infty} B_i = H_i$, where B_i is an approximation of H_i at ith iteration round. One widely used iterative process is Broyden, Fletcer, Goldfarb & Shanno method, named after its inventors. In this process B_i is the sum of two positive definite matrices

[2.21 a,b]

$$x_{i+1} = x_i - aB_i\nabla\widetilde{H}(\widetilde{x}_i)$$

$$B_{i+1} = B_i + \left(I + \frac{y_i^T B_i y_i}{s_i^T y_i}\right)\frac{s_i s_i^T}{s_i^T y_i} - \frac{s_i y_i^T B_i + B_i y_i s_i^T}{s_i^T y_i}$$

where $s_i = x_{i+1} - x_i$, $y_i = \nabla\widetilde{H}(\widetilde{x}_{i+1}) - \nabla\widetilde{H}(\widetilde{x}_i)$, $\widetilde{H}(\widetilde{x}_i)$ is defined in equation [2.15 b] and where B_0 and I are identity matrices. However, this iterative process contains so many matrix and vector multiplications that it does not represent any real time-saving in our optimal growth model with 15-dimension matrix inversion.

The tolerance range of the initial values is the main criterion for choosing the algorithm for the discrete dynamic general equilibrium models. With respect to the tolerance range, the multiple shooting method can be made a little more efficient than the bounded shooting method by increasing the number of subintervals. However, this also increases the dimension of the H-matrix.

After studying all three shooting algorithms we would still recommend the bounded shooting algorithm, provided that some prior knowledge exists, e.g. the absolute values of the initial and the final steady states of the dynamic system.

The following figure illustrates the performances of the bounded and the multiple shooting methods. Figures 2.5 and 2.6 show the bounded shooting method with an initial value of 4 and where the lower limit of the auxiliary boundary condition is -0.1, after 10 and 30 iteration rounds respectively. Figures 2.7 and 2.8 show are the multiple shooting method with all the initial values 10% lower than the solution and with the inverse Hessian matrix updated in every second iteration round, after two and six iteration rounds respectively.

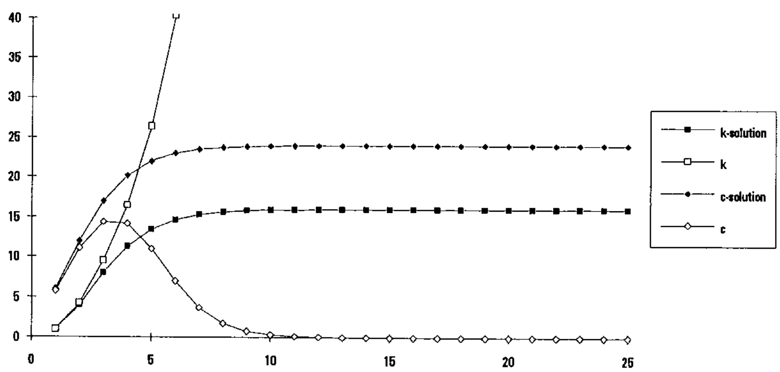

Figure 2.5: The bounded shooting method after 10 iteration rounds.

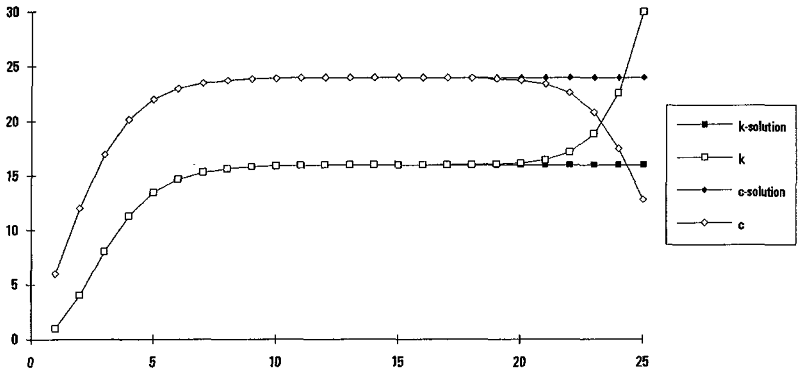

Figure 2.6: The bounded shooting method after 30 iteration rounds.

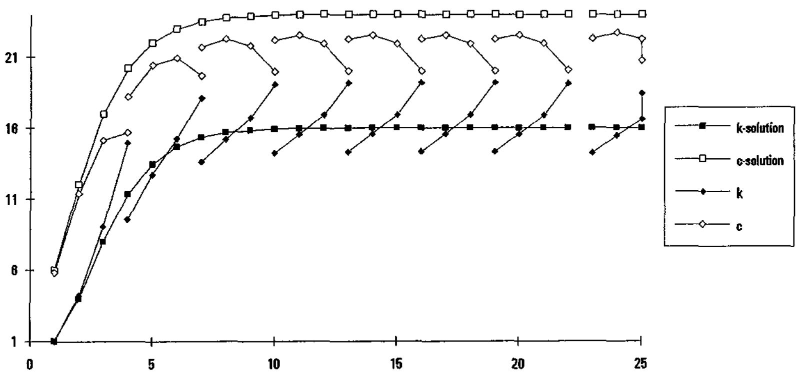

Figure 2.7: The bounded shooting method after two iteration rounds.

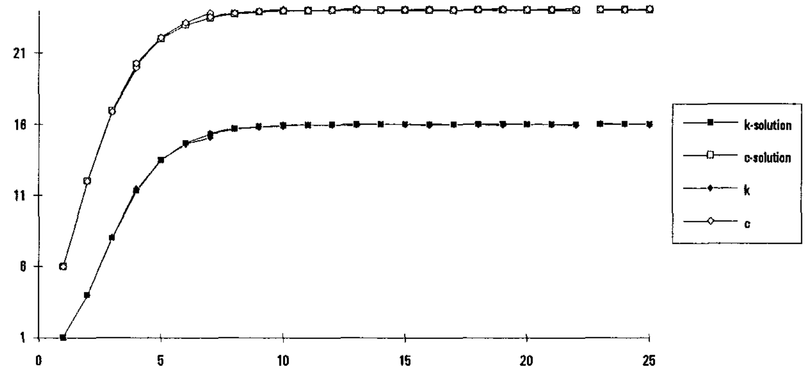

Figure 2.8: The multiple shooting method after six iteration rounds.

2.3 Fair-Taylor Methods

There are two Fair-Taylor methods described in literature. The first is the normal Fair-Taylor method, which was introduced by Fair and Taylor in 1983 and the second is the generalised Fair-Taylor method introduced by Wilcoxen in 1989. In the normal Fair-Taylor method algebraic methods alone are used to obtain a set of equations for fixed point iteration. The fixed point iteration is then performed using either the Gauss-Seidel or the Jacobi iteration method to achieve convergence. Wilcoxen's Fair-Taylor method improves the convergence by using calculus and Taylor approximation to obtain a set of equations for the fixed point iteration. Otherwise the methods are similar.

The Jacobi and Gauss-Seidel iterations are defined as follows. Consider an iterative process $x^{i+1} = f(x^i)$ Let x^n denote the n^{th} variable of x and let f_n denote the n^{th} component of the function f. Then we can write Jacobi iteration as

$$x_n^{i+1} = f_n(x^i, \ldots, x_m^i) \qquad n=1,\ldots,m$$

and Gauss-Seidel iteration as:

$$x_n^{i+1} = f_n(x^{i+1}, \ldots, x_{n-1,}^{i+1} x_n^i, \ldots, x_m^i), \quad n=1,\ldots,m$$

Thus Gauss-Seidel iteration uses the latest information available and should therefore sometimes converge faster than the corresponding Jacobi-algorithm. This is proven in Bertsekas & Tsitsiklis (1989). Under normal circumstances, the cost of using the Gauss-Seidel iteration represents a small additional programming effort.

2.3.1 Algorithm for the Normal Fair-Taylor method

In this section we use "hat" notation similar to the multiple shooting method. We define $\hat{c}^i$ as an input vector for the i^{th} iteration round and c^i is the outcome values of the i^{th} iteration round. The input vector for the $(i+1)^{th}$ iteration round, denoted as $\hat{c}^{i+1}$, is a linear combination of c^i and $\hat{c}^i$.

The normal Fair-Taylor method we describe here is designed to solve the system of difference equations in the optimal growth model:

$$k_{t+1} = F_1(k_t, c_t)$$

[2.5 a,b]

$$c_{t+1} = F_2(k_t, c_t).$$

In more complicated models market-clearing conditions must also be satisfied. Such conditions usually include equations that imply that time spent on labour and time spent on leisure must add up to all time in each time period. We will discuss these conditions in the next chapter.

The solution in the normal Fair-Taylor method begins by writing the Euler equation of the control variable[9] [2.5 b] in the following form:

[2.5 b'] $c_t = F_3(k_{t+1}, c_{t+1})$

Thus we want to have the values of the control variables at period t as a function of the control and state variables at period t+1. We do this because we rather to iterate new values for c_1 than for c_T. Secondly we make a guess for all expected variables $\hat{c}^0$. Thirdly we use [2.5 a], the guessed variables $\hat{c}^0$ and the knowledge of the initial values of k_0 to determinate the state variables k. Finally we use equation [2.5 b'] and the initial guesses for $\hat{c}^0$ and k to obtain new values for the control variables c^o. If and only if $\hat{c}^0 = c^o$, the values of the control variables satisfy the requirements for intertemporal optimal behaviour under the budget constraint [2.5 a]. Otherwise in the normal Fair-Taylor method we use a linear combination of $\hat{c}$ and c_t as a new set of initial values in the iteration process.

[2.22] $\hat{\mathbf{c}}^{i+1} = \varphi\mathbf{c}^i + (1 - \varphi)\hat{\mathbf{c}}^i$

If the convergence criterion is not met, we go back to step three of our iteration. With luck iteration converges to a fixed point [Keuschnigg (1991)]. If equations [2.22] and [2.5] form a contraction mapping in a closed subset $X \in R^n$, the iteration converges geometrically to unique solution from any initial vector $x_0(k,c) \in X(k,c)$ [Bertsekas & Tsitsiklis (1989)]. It is no easy task to verify contraction mapping due to the deep implicit structure of the equation system in the dynamic general equilibrium models.

The fourth step in the normal Fair-Taylor algorithm can be performed by applying either the Jacobi or the Gauss-Seidel method. In the Jacobi method the new value of c_t is a function of guessed values of the control variables at periods t and t+1.

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9 In economics literature control variables, especially related to the Fair-Taylor method, are referred to as jump variables. The name jump variable indicates that their values are not dictated by history as in the case of state variables.

$$c_t^i = F_3(\hat{c}_{t+1}^i, k_{t+1}^i(k_t^i, \hat{c}_t^i))$$

The structure of the Fair-Taylor method also allows us to use the Gauss-Seidel version with no significant increase in programming effort.

[2.5 b''] $\quad c_t^i = F_3(\hat{c}_{t+1}^{i+1}, k_{t+1}^i(k_t^i, \hat{c}_t^i))$

In the Gauss-Seidel version the new value of c_t is a function of guessed values of the control variables at period t and a ready-calculated value of the control variable at period t+1.

Using the normal Fair-Taylor method has both advantages and disadvantages. The first advantage is that the dimensions of the equation system to be solved are limited and the second is the absence of calculus. The costs of these advantages are increased computational time and increased uncertainty of convergence. Note that [2.5 b'] does not produce an update for the last time period. This has to be calculated by a separate final steady-state calculation or by adding an additional steady-state restriction $c_T=c_{T-1}$ as we did with the shooting algorithms. The third possibility is to fix a reasonable value for c_T and utilise the Turnpike property. The same applies to Wilcoxen's modified Fair-Taylor method. The flow diagram in the appendix illustrates Wilcoxen's modified Fair-Taylor method, but by letting the damping factors φ_1 and φ_2 be zero then the in the flow diagram illustrates the normal Fair-Taylor method.

2.3.2 Algorithm for the Wilcoxen's modified Fair-Taylor method

In Wilcoxen's modified Fair-Taylor method we perform the first three steps as in the normal Fair-Taylor method. We write the values of the control variables at period t as a function of the control and state variables at period t+1. We make a guess for all the expected variables $\hat{c}^0$. We use equation [2.5 a], the guessed variables $\hat{c}^0$ and knowledge of the initial state values k_0 to determine the state variables k. Thirdly, just as in the shooting methods, we apply the Taylor approximation of equation [2.5 b'] around the solution c^*_t. That leads an normal iteration formula similar to Newton's search:

[2.24] $$\mathbf{c}^i = (I - \nabla F_3)^{-1}(F_3 - \nabla F_3 \hat{\mathbf{c}}^i_t)$$

This equation has similar disadvantages to the multiple shooting method. Firstly the dimensions of the Jacobian matrix (n-m)T * (n-m)T are large. Secondly, at least in larger dynamic general equilibrium models, it is not very easy to calculate derivatives, due to their implicit structure. The Jacobian in equation [2.24] is defined as follows:

$$\nabla F_3 = \begin{bmatrix} J_{11} & J_{12} & \dots & J_{1T} \\ J_{12} & J_{22} & \dots & J_{2T} \\ \dots & \dots & \dots & \dots \\ J_{1T} & J_{T2} & \dots & J_{TT} \end{bmatrix} \quad J_{ts} = \begin{bmatrix} \frac{\partial c_{1t}}{\partial c_{1s}} & \frac{\partial c_{1t}}{\partial c_{2s}} & \dots & \frac{\partial c_{1t}}{\partial c_{(n-m)s}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial c_{(n-m)t}}{\partial c_{1s}} & \frac{\partial c_{(n-m)t}}{\partial c_{2s}} & \dots & \frac{\partial c_{(n-m)t}}{\partial c_{(n-m)s}} \end{bmatrix}$$

Wilcoxen also introduced some simplifying assumptions, which, according to Keuschnigg (1991), enormously reduce the computational cost without sacrificing a great deal of information about the intertemporal structure of the Jacobian of equation [2.24]. These assumptions are:

$J_{ts} = 0$ for $s > t+1$

$J_{ts} = 0$ $s < t$

$J_{tt} = J_{11}$ $t > 1$

$J_{tt+1} = J_{12}$ $t > 1$

The first assumption is straightforward. Varying $\hat{c}$ after period t+1 does not affect c in period t or earlier. This makes the Jacobian almost lower block triangular. The second assumption states that varying $\hat{c}$ in previous periods (s<t) does not affect c in period t or after. This is only an approximation, since the state equation [2.5 a] naturally carries over some effects into period t. This makes the Jacobian upper block triangular. Finally Wilcoxen assumes that the effect made to c by variation of $\hat{c}$ in periods t and t+1 is the same in all time periods and equal to J_{11} and J_{12}. This final assumption states that we only need to calculate two blocks of the Jacobian. After these assumptions we can rewrite equation [2.24] as:

[2.24'] $$c_t^i = (I - \varphi_1 J_{11})^{-1}(F_3(\hat{c}_{t+1}^{i+1}, \hat{c}_t^i) - \varphi_1 J_{11}\hat{c}_t^i + \varphi_2 J_{12}(c_{t+1}^i - \hat{c}_{s+1}^i))$$

This is the Gauss-Seidel version and the Jacobi version is quite similar. The φ symbols are dumping factors for the Jacobians. For values of c the upper index stands for the iteration round and the lower index stands for the time period. As in the normal Fair-Taylor method, the last step in Wilcoxen's modified Fair-Taylor method is to create a linear combination of $\hat{c}$ and c_t as a new set of initial values for the iteration process. If the convergence criterion is not met we go back to step three of our iteration. The flow diagram in the appendix illustrates Wilcoxen's modified Fair-Taylor method. Note that when using $j_1 = j_2 = 0$ in Wilcoxen's modified Fair-Taylor method, we have the normal Fair-Taylor method.

2.3.3 Performance of the Fair -Taylor Algorithms

We used the same optimal growth model as in the shooting method section to analyse the performance of the Fair-Taylor algorithms. For the iteration process the Gauss-Seidel version of the Euler equation [2.5 b'] becomes:

$$c_t = F_3(\hat{c}_{t+1}^{i+1}, k_{t+1}(k_t, \hat{c}_t^i)) = \frac{\hat{c}_{t+1}^{i+1}}{A\alpha\beta k_{t+1}(k_t, \hat{c}_t^i)^{\alpha-1}}$$

The Jacobi version of [2.5 b'] is otherwise similar, expect that $\hat{c}_{t+1}^{i+1}$ is replaced by $\hat{c}_t^i$.

The main difficulty in both of the Fair-Taylor algorithms lies in the initial guesses of the control variables ($\hat{c}$'s). If the guess in most of the time periods were to be higher or lower than the solution, this would lead to very high errors in the state variables (k). In the Fair-Taylor algorithms there are no mechanisms similar to, the auxiliary boundary condition in the bounded shooting method or the guesses in the state variables in the multiple shooting method to deal with this problem. This is why we added an extra auxiliary boundary condition in all our runs of the Fair-Taylor algorithms. We added the following constraint in equation [2.5 a], which reduced overflows a great deal.

[2.25 a']

$$k_{t+1} = Ak_t^{\alpha} - c_t \qquad \text{if } t<15$$

$$k_{t+1} = \mathrm{Max}(10, Ak_t^{\alpha} - c_t) \quad \text{if } t>1$$

Another aspect, which in Wilcoxen's modified Fair-Taylor method we did not find very beneficial for the optimal growth model, was the third and fourth simplifying assumptions. We found that at the beginning of the iteration process the Jacobians did vary quite a lot in different time periods. Before introducing the auxiliary boundary condition it also caused overflows. For example if every second initial guess is 1% over and every other initial guess is 1% less than the solution, the Jacobians vary as in picture 2.9. The picture implies at least that the fourth simplifying assumption, which states $J_{s,s+1}=J_{1,2}$ is not that good approximation. With our case it means that for example the $J_{20,21}$, which is 1.13, is approximated by 0.5, which is the value of $J_{1,2}$. The error caused by the third simplifying assumption in our case is quite small, because the values of $J_{s,s}$ are almost the same. However, in problems with large dimensions it is probably a good idea to use assumptions three and four, or perhaps to calculate the Jacobians for every n^{th} time period and use interpolation in between.

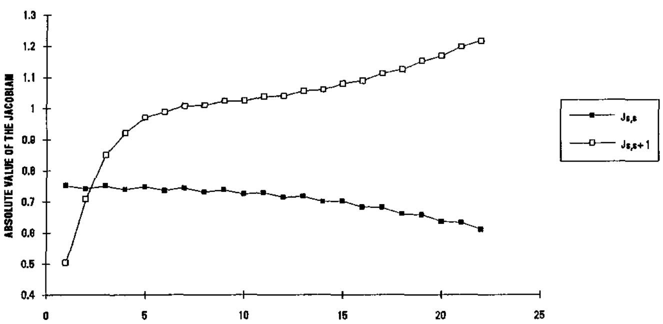

Figure 2.9: Absolute values of the Jacobians of the various time periods in the optimal growth model when the guesses for the consumption vector (c) are 1 % over the actual solution in every second and 1 % less the actual solution in every second time period.

To test convergence we ran the normal Fair-Taylor method and two versions of Wilcoxen's modified Fair-Taylor method. The first Wilcoxen's version uses Jacobi iteration and assumptions 1-4. The second Wilcoxen's version uses Gauss-Seidel iteration and assumptions 1-2 only, so we calculated Jacobians for each time period. We do not use damping factors for the Jacobians, so we had $\varphi_1 = \varphi_2 = 1$. We set the initial values so that guesses in odd time period were below the solution by a certain percentage and guesses in even time periods were over the solution the solution by certain percentage. The convergence criterion was $\sum_{t=1}^{T} (c_t - \hat{c}_t)/T < 1 * 10^{-6}$.

We chose the following relaxation parameter functions for the iteration results in table 2.4. For the normal Fair-Taylor method we had a minimum of 0.25 or $0.01*i^{1.2}$. Letter i stands for the iteration round number. The choice of relaxation parameter for both Wilcoxen's methods turned out to be more complicated.

Heuristically the Gauss-Seidel method should converge faster than the Jacobi method, but we obtained the opposite results when we used the same relaxation parameter function for both methods. The reason for this is probably purely accidental. The relaxation parameter function was accurate for the Jacobi method but too small for the Gauss-Seidel method in the few first iteration rounds and the Jacobi method reached the neighbourhood of the fixed point in fewer iterations. By using different relaxation parameter functions we were able to make the Gauss-Seidel iteration converge faster.

In the following table we used the following relaxation parameter functions. For the Jacobi method we had a minimum of 0.5 and 0.02*i and for the Gauss-Seidel method we had a minimum of 0.5 and 0.1*i. When we used the Jacobi relaxation parameter function for the Gauss-Seidel method the convergence was usually one iteration round slower compared to the Jacobi method.

Table 2.4: Required iteration rounds for convergence of the different Fair-Taylor algorithms. Criterion for convergence $\sum_{t=1}^{T} (c_t - \hat{c}_t)/T \quad < 1 * 10^{-6}$.

Error in initial values	Normal Fair-Taylor (Jacobi)	Wilcoxen's Fair-Taylor (Jacobi)	Wilcoxen's Fair-Taylor (Gauss-Seidel)
2%	68	55	52
4%	72	58	54
10%	76	61	55
20%	77	62	62

Table 2.4 indicates that the Wilcoxen's modified methods do not offer any significant advantage compared to the normal Fair-Taylor algorithm. It is therefore sensible to try the normal Fair-Taylor method first. At least in problems with small

dimensions, if the normal Fair-Taylor method fails to converge the algorithm written for it can be used to write Wilcoxen's Fair-Taylor algorithm.

In our opinion that name the normal Fair-Taylor algorithm is somewhat surprising. In most respects we found it to be Jacobi or Gauss-Seidel fixed point iteration method.

2.4 Comparison of the performances of the shooting and the Fair-Taylor algorithms

Comparison of the bounded and multiple shooting methods with Wilcoxen's Fair-Taylor method in the case of models such as the optimal growth model favours the shooting methods.

We discuss overlapping generations models in the next chapter. In these models the optimal intertemporal relations in the control variables are not so simple, as we will see. Therefore the shooting methods are very difficult to apply. The normal Fair-Taylor method also has some other advantages compared to the shooting methods: it is easy to apply and it converges quite often, at least after some work with the initial guesses. It is also widely used, for example Auerbach & Kotlikoff (1987) arrange their equations of the optimal behaviour more or less as suggested by the Fair-Taylor method and use Gauss-Seidel iteration method to solve the system of equations. However they themselves state that they apply the Gauss-Seidel method and not the Fair-Taylor method. Thus we do not attempt to differentiate strictly between the Gauss-Seidel and the normal Fair-Taylor method here.

After applying all these methods we have come to the conclusion that for calculating the transition paths between steady states the plain Gauss-Seidel or Jacobi methods with a relaxation parameter are those to be used first. The equations to be iterated are usually contraction mappings, at least when a relaxation parameter is used. In this way these iteration methods should converge. If these methods fail to converge, we advocate leaving the equations as suggested by the normal Fair-Taylor method and replacing the Gauss-Seidel or the Jacobi method by Newton's or gradient methods.

3 TWO LARGE-SCALE DYNAMIC GENERAL EQUILIBRIUM MODELS

In the preceding chapter we discussed several numerical methods for solving the optimal growth model. In this chapter we replace the optimal growth model by two more sophisticated models[10]. We call these models the long-lived agents model and the overlapping generations model. These models separate households and firms and allow separate payments to the owners of labour and capital. In addition to calling these models the long-lived agents model and the overlapping generations model these names are also general names for different structures for modelling the life time behaviour [Sargent (1987)] and economic growth of households mathematically [Azariadis (1993)] and [Barro & Sala-i-Martin (1994)]. Both these structures have been widely used in economic textbooks but policy simulation models of a particular country usually applies the overlapping generations structure.

We have divided this chapter into five parts. In the first subsection we briefly describe the long-lived agents model. In the second subsection we formulate dynamic first-order conditions of optimal behaviour. They include the initial and the final steady states and the transition path between them. Thirdly we analyse how technological innovation affects wages and interest rates, the demand for labour, the utility of households and so on. We do this by solving the model using the normal Fair-Taylor method with the Jacobi iteration method. Fourthly we describe differences between the long-lived agents model and the overlapping generations model and derive the first-order conditions of this model. Finally we solve a policy simulation with the overlapping generations model using similar technological innovation as in the long-lived agents model. We use Jacobi iteration to solve the overlapping generations model.

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[10] The models in this chapter are self-designed and based on the model used by Auerbach and Kotlikoff in "Dynamic Fiscal Policy".

3.1 Description of the long-lived agents model

Consider a non-stochastic, discrete-time, one-good production economy consisting of a large constant amount of households and firms. The firms use capital and labour as inputs to produce the good that can be used for consumption or investment. The households are the economic agents in this model because in addition to their own economic actions they run the firms as the owners. All households live identical T periods. Furthermore we assume that all households are born and die at the same time. We also let T run to infinity. So the time horizon is identical to the optimal growth model and we avoid formulation of the "end of the world".

There is no public sector in the economy we model. The economy is also closed, so there are no foreign countries to do trade with. The economy is at the (initial) steady state. At the beginning of the first period, a technological innovation changes the production function in such a way that the production process becomes more capital-intensive and less labour-intensive, the other input good for production. Simultaneously production becomes more efficient. We use this model to study the consequences of this innovation.

3.1.1 Household behaviour

The households are endowed with 24 hours of leisure each day. They can intertemporally exchange their leisure for the consumer good by going to work to earn the amount of the good they want to consume. Each household makes economic actions so that it maximises its utility over its life-time. The utility is an intertemporal function of consumption abilities and leisure. Each household has its own utility function, but to be able to solve the model numerically we must replace the households by a single average household, which behaves as if there were more household in the economy [Sargent (1987)] and [Perraudin & Pujol (1991)].

Thus the household maximises by exchanging intertemporally its endowed leisure (l) for the consumption good (c) so that its intertemporal utility achieves the maximum value. The household's utility function takes the following form:

$$U = \frac{1}{(1 - 1/\alpha)} \sum_{t=1}^{T} \frac{1}{(1 - 1/\delta)^{(t-1)}} u_t^{\frac{(1-1/\alpha)}{(1-1/\rho)}} \qquad [3.1]$$

where u_t is defined as

$$u_t = (c_t^{(1-1/\rho)} + \alpha_0 l_t^{(1-1/\rho)}).$$

This is the normal CES function [Chiang (1984)] and [Varian(1992)] of consumption and leisure, where c_t and l_t are respectively consumption and leisure in period t. Symbols a ,a_0, d and r are taste parameters that allow of several different tastes and individual behaviour to be presented. In the model they are exogenous and therefore we call them exogenous parameters. These parameters should be estimated from the data of the economy we aim to model. The discount rate d represents the pure time preference of the households. The smaller the value of d, the less the household will consume now compared to future. We will discuss the effects of a, a_0 and r when deriving the first-order conditions for the household.

More complicated forms of household utility functions could include the disaggregation of the consumption goods. However, this complicates the solution method and in particular the parameter estimations quite considerably.

To avoid infinite borrowing, so referred as the Ponzi-game [Barro & Sala-i-Martin (1994)] in economic literature, we introduce a budget constraint. Since there is no social security or taxes, the budget constraint simply prevents households from consuming more during their lifetime than their total income from labour and capital.

So the present value of consumption cannot exceed the sum of the present value of earning and the present value of capital gain.

[3.2] $$\sum_{t=1}^{T} \frac{w_t(1-l_t)-c_t}{\prod_{s=2}^{t}(1+r_s)} = 0$$

where r_t and w_t are respectively the interest rate and wage rate in period t. Because household has no bequest motive and because of the structure of the CES utility function:

$$\frac{\partial U}{\partial c} > 0 \wedge \frac{\partial U}{\partial l} > 0 \ \forall \ c \geq 0 \wedge l \geq 0$$

optimal behaviour is to consume everything before dying, hence the equality sign in equation [3.2]. More binding forms of budget constraint can prevent households from borrowing against their future income. And if the public sector existed, it would include taxes and social security.

We must also scale leisure so that there cannot be a negative labour supply. So leisure and labour time must add up to all time (scaled as one time unit) in each period.

[3.3 a] $$l_t \leq 1 \qquad \forall t$$

[3.3 b] $$l_t + L_t = 1 \qquad \forall t$$

The question as to how long each period is cannot be answered with accuracy in models that use the long-lived agents model structure because household has an infinite life horizon [Barro & Sala-i-Martin (1994)] and [Sargent (1987)].

3.1.2 Optimal behaviour of the firms

Both in the long-lived agents model, and the overlapping generations model we have one production sector and a closed economy. By making the first assumption we indicate that same good can be either consumed or invested. By making the second assumption we avoid the formulation of exports, imports and exchange rates. We also assume that all goods produced in period t must be consumed or invested in the same period. The firms operate in a competitive manner producing a good, for which there is demand from the households. In the production sector we do a similar aggregation to the household sector. We let one average competitive firm represent all the firms. The firm produces the single good by using labour and capital (the goods invested), which are supplied by the household. The prices the firm pays to the household for labour and capital are determined endogenously.

We use a CES production function similar to the household's utility function to calculate the output of the firm as a function of two inputs: capital (K) and labour (L). The CES production function takes the following form:

[3.4] $$F_t = \varepsilon_1(\varepsilon_0 K_t^{(1-1/\varepsilon)} + (1-\varepsilon_0)L_t^{(1-1/\varepsilon)})^{\frac{1}{1-1/\varepsilon}}$$

where $F_{t,}$, $K_{t,}$ and $L_{t,}$ are output, capital and labour at period t respectively. e_1 is a scaling constant. The technological progress we simulate means that at the beginning of the period one technological innovation increases e_1, and simultaneously production becomes more capital-insensive, which increases e_0. The effects of the parameters e_0 and e on output will be discussed later in this chapter.

The profit of the firm in period t is:

[3.5] $$\pi_t = F_t - w_t L_t - r_t K_t,$$

where w_t and r_t are the wage level and interest rate in period t respectively. In a fully competitive economy the firm's profits (p_t) are zero in each period. In brief this is because if the competitive firm tries to maintain above market prices nobody will buy its output and if it tries to pay below market for its inputs, nobody will work for it or lend capital to it [Varian (1992)] and [Barro & Sala-i-Martin (1994)]. We set the price of the output good as the numeraire. Thus the price of capital and labour are calculated as a function of the price of the output good in each period.

Competition requires that employees are paid according to their marginal product. This allows us to derive the wage level from [6a] $\frac{\partial \pi_t}{\partial L_t} = 0$ in each period t. Equation [6.a] implies that the firm will hire more labour (by paying higher wages) as long as the additional labour produces only the value the firm must pay for it[11]. The assumption of a fully competitive economy also requires that firms can adjust their labour without an additional cost and that markets are fully competitive. In other words it means that each employee negotiates his wage without any restrictions such as a minimum wage. In this case all leisure time is voluntary and there is no unemployment. Thus each individual is able to supply the type of labour demanded by the firms without education.

We can derive the interest rate in the same manner as the wage rate. The firm can adjust its capital without an additional cost and there is no additional user fee of capital. Thus the case interest rate r_t in each period is the solution of the marginal productivity of the capital to the firm [6b] $\frac{\partial \pi_t}{\partial K_t} = 0$.

---

[11] Because the CES production function [3.4] is concave in respect of labour (L) (and also in respect of capital when we speak of the interest rate) this point exists.

3.1.3 Market equilibria

To satisfy the market equilibria, production in each period must be equal to consumption plus investment in that period. So we do not model any exogenous depreciation rate.

[3.7] $$K_{t+1} = K_t + F(K_t, L_t) - c_t$$

3.2 The Kuhn-Tucker necessary and sufficient conditions for the long-lived agents model and the role of the exogenous parameters in the utility and production functions

In this subsection we derive the necessary and sufficient conditions for optimal behaviour in the long-lived agent model under conditions of perfect foresight. We do this by using the Kuhn-Tucker conditions, which differ slightly from the recursive dynamic optimisation technique we used to solve the optimal growth model in the preceding chapter.

3.2.1 The household

We maximise [3.1] with the budget condition [3.2] and the leisure condition [3.3 a] by writing the Lagrangian function and applying the Kuhn-Tucker necessary conditions for optimality [Bazaara & Shetty (1979)]:

Min -U

subject to $$\sum_{t=1}^{T} \frac{w_t (1 - l_t) - p_t c_t}{\prod_{s=2}^{t} (1 + r_s)} = 0$$

and $$l_t - 1 \leq 0$$

The Kuhn-Tucker necessary conditions[12] for optimality are:

[3.7] $$\nabla(-U(C_t, l_t)) + \lambda \nabla BC(C_t, l_t) + \mu_t \nabla LB(C_t, l_t) = 0$$

[3.8] $$\mu_t LB(C_t, l_t) = 0$$

[3.9] $$\mu_t \geq 0\ ,$$

where BC, LC, λ and μ_t are respectively the budget and leisure constraints and Lagrange multipliers associated to them. The Lagrange multipliers are also referred to respectively as the shadow price of the lifetime budget constraint and the shadow wage. In addition to the necessary conditions, the Kuhn-Tucker sufficient conditions [Bazaara & Shetty (1979)] are also satisfied, because the CES functions are quasi-concave (-U is quasi-convex) with all positive values of capital and labour [Chiang (1981) pp.427], and because both constraints are linear both in terms of C_t and l_t. Mathematically the budget constraint is not linear in terms of l_t, because the wage rate is a function of leisure, $w_t=w_t(K_t,L_t(l_t))$, but the household does not take into account in its optimisation problem the fact that its supply of labour effects its wage rate. In this way we prevent the single household from behaving as a monopoly in selling labour and capital.

After a little manipulation equation [3.7] attains the forms:

[3.7 a] $$\frac{1}{(1-1/\delta)^{(t-1)}} u_t^{\frac{(1-1/\alpha)}{(1-1/\rho)}} c_t^{-1/\rho} = -\lambda\left(\frac{1}{\prod_{s=2}^{t}(1+r_s)}\right)$$

[3.7 b] $$\frac{1}{(1-1/\delta)^{(t-1)}} u_t^{\frac{(1-1/\alpha)}{(1-1/\rho)}} \alpha_0 l_t^{-1/\rho} = -\lambda\left(\frac{w^*}{\prod_{s=2}^{t}(1+r_s)}\right)$$

where w* is defined as:

---

[12] In economics literature necessary conditions are usually first-order conditions and similarly sufficient conditions are second-order conditions.

$$w_t^* = w_t + \frac{\mu_t \prod_{s=2}^{t} (1 + r_s)}{\lambda}.$$

Dividing equation [3.7 a] by [3.7 b] leads to an equation of the optimal relation of contemporaneous consumption and leisure:

[3.11] $$\frac{C_t}{l_t} = \left(\frac{\alpha_0}{w_t^*}\right)^{-\rho}.$$

Equation [3.11] clarifies the meaning of two the taste parameters α_0 and ρ. The parameter ρ determines how responsive the labour supply is to real wage of the same period. The term α_0 represents the preferences of the household to leisure relative to consumption. An increase in α_0 increases l and if α_0 is held fixed, the percentage change in the leisure consumption ratio, l_t/c_t, in respect of change in the wage rate is equal to ρ. Equation [3.11] allows us to use one control variable only - consumption - and calculate optimal leisure from [3.11]. Note that this is only valid if we have an interior solution in respect of leisure, thus $l_t < 1$ in all periods.

The consumption path is derived by replacing l_t in [3.7 a] by [3.11] and dividing [3.7 a] by itself lagged one time period:

[3.12] $$\frac{C_t}{C_{t-1}} = \left(\frac{1 + \alpha_0^{\rho}(w_t^*)^{-\rho+1}}{1 + \alpha_0^{\rho}(w_{t-1}^*)^{-\rho+1}}\right)^{\frac{\rho-\alpha}{\rho-1}} \left(\frac{(1 + r_t)}{(1 + \delta)}\right)^{\alpha},$$

and for leisure by replacing consumption variables (c) in equation [3.12] by the values of [3.11].

[3.13] $$\frac{l_t}{l_{t-1}} = \left(\frac{1 + \alpha_0^{\rho} w_t^{*\ -\rho+1}}{1 + \alpha_0^{\rho} w_{t-1}^{*\ -\rho+1}}\right)^{\frac{\rho-\alpha}{\rho-1}} \left(\frac{(1 + r_t)}{(1 + \delta)}\right)^{-\alpha} \left(\frac{w_t^*}{w_{t-1}^*}\right)^{-\rho}$$

Equation [3.12] shows the effect of the last remaining taste parameter α. This represents the intertemporal elasticity of substitution of consumption across various periods of the household's life. For example behaviour of the household related to a real wage rate increase in time causes two things. First consumption increases over time and secondly leisure decreases during lifetime, because the household shifts its labour supply to later years to take advantage of higher wages.

Note that in this chapter we derived the first order conditions by maximising the utility function [3.1] subject to the budget constraint [3.2] and we applied the Kuhn-Tucker optimality equations to the Lagrange equation. We would have had the exact first order conditions, if we had maximised utility [3.1] with respect to market clearing equation [3.7] as a costate equation and had applied the first-order conditions of discrete recursive dynamic optimisation [Kendrick (1981)] and [Kirk (1970)], as we did with the optimal growth model in chapter 2.

3.2.2 The firm

The fully competitive markets force the firm to pay wages and interest according to solutions of equations [3.6 a] and [3.6 b], which are:

[3.14] $$w_t = \varepsilon_1(1-\varepsilon_0)(\varepsilon_0 K_t^{(1-1/\varepsilon)} + (1-\varepsilon_0)L_t^{(1-1/\varepsilon)})^{\frac{1}{\varepsilon-1}}$$

[3.15] $$r_t = \varepsilon_1\varepsilon_0(\varepsilon_0 K_t^{(1-1/\varepsilon)} + (1-\varepsilon_0)L_t^{(1-1/\varepsilon)})^{\frac{1}{\varepsilon-1}}.$$

Note that the household considers w_t and r_t to be given constants, because it "thinks" that there are a large number of households and the household considers that its own labour and savings decisions do not effect wage or interest rates.

To understand the meaning of the parameters ε, ε_0 and ε_1 it is useful to maximise [3.5] in respect of K_t and L_t without replacing w_t and r_t. This leads to a relation similar to [3.11]:

[3.16] $$\frac{L_t}{K_t} = \left(\frac{r_t(1-\varepsilon_0)}{w_t\varepsilon_0}\right)^{\varepsilon}.$$

As with the contemporous consumption leisure equation [3.11], we can see from the contemporous capital labour equation of the firm [3.16] that ε is the elasticity of substitution in production, representing the percentage change in the ratio of K_t/L_t in respect of a percentage change in the wage/interest ratio. If the interest/wage ratio rises, the firm is more willing to employ. In reality firms also face costs in adjusting their capital and labour as well. We have not included this fact in these equations.

The parameter ε_0 represents the intensity of capital use in production. The higher ε_0 is, the smaller the labour-capital ratio the firm has. The parameter ε_1 is a scaling constant.

3.3 Technological innovation: policy simulation with our long-lived agents model.

3.3.1 Description of the algorithm for the numerical solution

We solved the model in three parts. The first part was the initial steady state, the second the final steady state and the third the transition path in between. This division is very common in economic literature.

In the initial and final steady states the derivatives of K, L and c are zero. So we had to iterate K, L, c and μ to find a solution to the following fixed point equation system.

$$c_t = \left(\frac{1 + \alpha_0^{\rho}(w_t^*)^{-\rho+1}}{1 + \alpha_0^{\rho}(w_{t-1}^*)^{-\rho+1}} \right)^{\frac{\rho-\alpha}{\rho-1}} \left(\frac{(1 + r_t)}{(1 + \delta)} \right)^{\alpha} c_t$$

$$1 - L_t = l_t = \left(\frac{\alpha_0}{w_t^*} \right)^{\rho} c_t$$

[3.17 a,b,c,d]

$$K_t = K_t + F(K_t, L_t) - c_t$$

$$\mu_t(l_t - 1) = 0 \ .$$

We solved the steady states in the following way. We set the exogenous parameters to the values before a technological innovation and assume $l_t < 1$ for all periods t. According to the Kuhn-Tucker theorem this implies that all the Lagrange multipliers (μ_t) associated with inequality constraints $l_t < 1$ are zero. Next we guess K, L and c. Then we used normal Jacobi fixed point iteration to achieve convergence. After each iteration round we checked, whether our assumption $l_t < 1$ held, and if not the iteration was terminated. The same iteration process was performed with the exogenous parameters after the technological change.

We solved the transition path with knowledge of the value of capital in the initial steady state and the values of consumption and the labour supply in the final steady state. Again we assume that $l_t < 1$. We set T at 40 periods. We guessed the values of c and L from period 1 to T-1. Then we solved the values of w, c and K, note $K_0 = K_1$, using equations [3.14],[3.15] and [3.7].

Finally we apply equations [3.11] and [3.12] to achieve the next round of guesses for the values of c and L. In every iteration round we also checked that l_t were less than one for every period, so that our assumption of the interior solution, $\mu_t = 0$, would hold during the iteration process.

For the initial steady-state calculations it takes about 50 iterations to converge in terms of eight decimals. The final steady state only takes about 30 iterations to converge because we can utilise the initial steady state values as initial guesses and we used a higher relaxation parameter. When solving the transition path we used the final steady state values as initial guesses, with the exception of the first few periods, where we use linear combinations of initial and final steady state-values. We used a very small constant relaxation parameter. We found the vectors **K**, **c** and **l** that satisfied all conditions of optimality after very many iteration rounds, but even in the Windows environment this was not very time consuming. We will discuss the uniqueness and stability of the steady states and the transition path in the next chapter.

3.3.2 The simulation results of a technological innovation with the long lived agents model

We use the following parameters (table 3.1) in the model. With the exception of the scaling constant they are those estimated by Kenc et al. (1994) for Finland using micro- and macroeconomic time series data for Finland dating from the early 1960s to the early 1990s. Just to take one example, the regulations that limited households from borrowing from the banks have changed entirely between the years 1960 and 1992, so these exogenous parameters must have "fairy" high confidence intervals. The confidence intervals, however, were not included in Kenc et al. (1994).

Table 3.1: The parameter values of the long-lived agents model.

Parameters of the household	Value	Parameters of the firms	Value (before technological innovation)	Value (after technological innovation)
α_0	4	ε	0	0
α	1	ε_0	1	1
ρ	1	ε_1	1	1
δ	0			

The technological innovation increases utility immediately, because the innovation increases ε_1, which can be utilised immediately with existing capital and labour. Figure 3.1 shows how utility increases during the transition path. The increase in utility after the initial "shock" comes mainly from changes in ε_0. Leisure increases as the capital/labour ratio converges to a new steady state. The main difference for computable static models[13] can be seen in the transition path of the interest rate. This jump is not visible in static models. To really benefit from time spent solving transition paths in order to analyse how, for example, the technological innovation affects to the utility of households born at different periods, we should use the overlapping generation model structure.

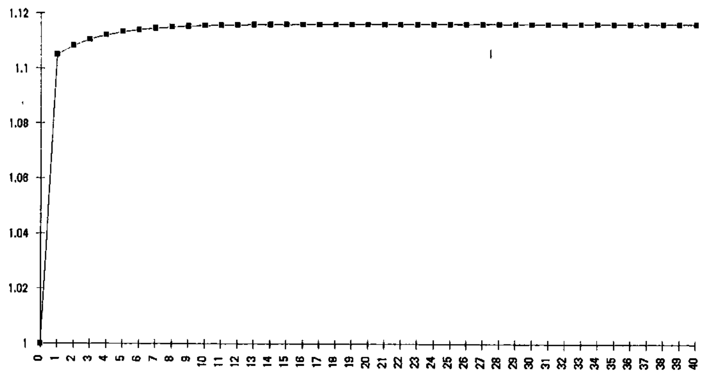

Figure 3.1: Development of utility without discounting to period zero. Initial steady state = 0.

13 A model where only initial and final steady state solution are calculated.

The other variables converge to the final steady state, as figure 3.2 shows.

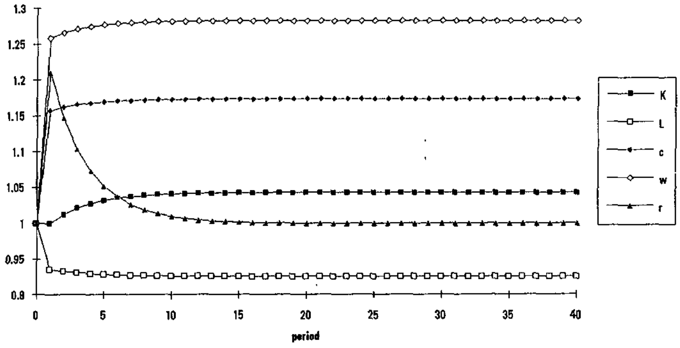

Figure 3.2: Transition path of state-, control variables, labour supply and wage and interest rates.

3.4 The overlapping generations model

The overlapping generations model brings the mathematical relations of the long-lived agents model closer to reality by letting new households be born and especially die. Thus at any period there are individuals of different ages living in the same economy. The idea of overlapping generations was introduced by Diamond (1965) and Samuelson (1957).

The concept of overlapping generations is a key structure that permits the study of pay-as-you-go[14] pension systems for example. Working generations accept the system because they trust that they will be paid when they will become old. The overlapping generations model is a good tool for analysing whether this trust is well founded. The trust may be called in question if the population or the economy is not growing or the economy is borrowing major amounts of its current spending from foreign economies.

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[14] Pension payments by current working generations are paid to current old generations instead of accruing for the payers themselves.

For the model period 1 is referred to as the period where the initial steady state is breached. Generation 1 is born in period 1 and so on. Figure 3.3 illustrates the overlapping generation structure. In each time period one generation is born and one dies. Thus each generation optimises its own utility without concerning itself without other generations.

	1	2	3	4	time period	╲
1	generation 1		generation 3			╱
2		generation1		generation 3		.....infinity
3			generation1		generation 3	.....infinity
4	generation -2			generation1		.....infinity
5		generation -2			generation1	.....infinity
age:			generation-2			.....infinity
periods				generation-2		.....infinity
╲	╱	death	death	death	death	death

Figure 3.3: Illustration of the age-time dimensions of the models with overlapping generations structure. In period 3, for example, the generation 3,2,1,0,-1,-2,-3 are alive.

In this chapter we describe a simple overlapping generations model and study how equations of first-order conditions must be arranged to be able to use the Jacobi iteration method described earlier.

3.4.1 Description of the overlapping generations model

Assume that in each period one unit of households is born and each household lives two periods. No household has initial capital, implying that the households do not leave any bequests. We use the same variables as in the long-lived agents model except that we add one more variable, namely labour efficiency, e, which is a function of individuals' age in terms of periods, not time. The labour efficiency variable permits young individuals to have different labour productivity from that of old ones.

We also add a lower index to show, which generation the variable or the parameter refers to in any period. For example $c_{t,old}$ refers to the consumption of the old generation living in period t. After introducing the labour efficiency variable, e, and households that die after two periods, the maximisation problem of the household in the long-lived agents model becomes to the maximisation problem of the mortal households in the overlapping generations model. The exogenous taste parameters and the exogenous parameters in the production function have the same meaning as in the long-lived agents model.

[3.18] $$\text{Max} \quad U = \frac{1}{(1-1/\alpha)} \sum_{t=1}^{2} \frac{1}{(1-1/\delta)^{(t-1)}} u_t^{\frac{(1-1/\alpha)}{(1-1/\rho)}}$$

where u_t is defined as

$$u_t = (c_t^{(1-1/\rho)} + \alpha_0 l_t^{(1-1/\rho)}) \ .$$

We define the budget constraint so that economic transactions take place at the beginning of each period. We can write the budget constraint for each generation in the following way:

[3.19] $$(1+r_s)(w_t e_{t,young}(1-l_{t,young}) - c_{t,young}) + (w_t e_{t,old}(1-l_{t,old}) - c_{t,old}) = 0$$

Thus each generation has its own optimisation problem to maximise [3.18] under the budget constraint [3.19] and the inequality constraint for leisure [3.3.a] $l_t < 1$ for all periods.

The production sector is identical to that in the long-lived agents model. The firm does not die, but the owners do. Thus the optimisation problem of the firm is identical to that of our long-lived agents model. The results determine the wage rate and the interest rate. The only difference is in the definition of the labour input. In the long-lived agents model it was $L_t=1-l_t$. In the overlapping generations model it is: $L_t=e_{young}*(1-l_{t,young})+e_{old}*(1-l_{t,old})$.

3.4.2 The Kuhn-Tucker necessary and sufficient conditions for optimality in the overlapping generations model

Because the households only live fir two periods the recursive optimisation method is of no benefit. The normal Kuhn-Tucker method gives the optimal consumption path. It is very similar to long lived agents model.

$$[3.20] \qquad \frac{C_{t,old}}{C_{t-1,young}} = \left(\frac{1+\alpha_0^{\rho}(w^*_{t,old})^{-\rho+1}}{1+\alpha_0^{\rho}(w^*_{t-1,young})^{-\rho+1}} \right)^{\frac{\rho-\alpha}{\rho-1}} \left(\frac{(1+r_t)}{(1+\delta)} \right)^{\alpha},$$

where w* is now defined

$$w_t^* = w_t e_s + \frac{\mu_t(1+r_s)}{\lambda}.$$

When we solve the model by assuming the interior solution with regard to the leisure constraint $l_t<1$, all the Lagrange multipliers (μ_t's) are zero. Equation [3.20] implies that the consumption of the young at period t-1 is related to the consumption of the old at period t. This causes problems if we use one of the shooting methods. If the household lived for more periods, it might be a sensible idea to use shooting methods, where each generation's control variable, the consumption, is a separate dimension.

In steady states consumption is only a function of the age period, and not of the time period. In steady states the consumption of the old generation is a function of the consumption of the young generation alive in the same period. The same generational relation is naturally true for the steady state leisure value.

We obtain the optimal ratio of contemporaneous leisure and consumption as we did in the long-lived agents model. The only difference is the addition of labour efficiency.

[3.21] $$l_{t,s} = \left(\frac{\alpha_0}{e_s * w_{t,s}}\right)^{\rho} c_{t,s} \text{ (interior solution)}$$

where the index s can either mean young or old age. In our overlapping generations model the young generation alive inherits no capital, so the capital of the economy is owned by the old generation alive. The old generation alive, however, spends its capital before it dies, and all the input capital for the firm at period t must be saved by the young generation alive at period t-1. The market-clearing equation for our overlapping generations model is:

[3.22] $$K_{t+1} = (1 + r_t) * ((1 - l_{t,young}) * e_{young} * w_{t,young} - c_t)$$

Owing to the same CES function structures as in our long-lived agents model, the sufficient conditions are also satisfied, as they were in the long-lived agents model.

3.4.3 The steady-states

The following equations must be satisfied in the steady-states of the overlapping generations model. The derivative of the capital must zero, implying that the young generation alive must save an amount of capital identical to that owned by the old generation alive. So at the steady states the market clearing equation is

[3.23 a] $K_t = (1 + r_t) * ((1 - l_{t,young}) * e_{young} * w_t - c_t)$

The consumption of the young generation alive and the old generation alive and the ratio of contemporaneous leisure and consumption of both age groups must satisfy the optimality conditions of the maximisation problem of the household.

$$\frac{c_{t,old}}{c_{t,young}} = \left(\frac{1 + \alpha_0^{\rho}(w^*_{t,old})^{-\rho+1}}{1 + \alpha_0^{\rho}(w^*_{t,young})^{-\rho+1}}\right)^{\frac{\rho-\alpha}{\rho-1}} \left(\frac{(1 + r_t)}{(1 + \delta)}\right)^{\alpha}$$

[3.23 b,c,d] $$\frac{c_{t,young}}{l_{t,young}} = \left(\frac{\alpha_0}{w^*_{t.young}}\right)^{-\rho}$$

$$\frac{c_{t,old}}{l_{t,old}} = \left(\frac{\alpha_0}{w^*_{t,old}}\right)^{-\rho}$$

The budget constraint for each household must also be met:

[3.23 e] $(1 + r_t)((1 - l_{t,young})e_{young}w_t - c_{t,young}) + (1 - l_{t,old})e_{old}w_t - c_{t,old} = 0$

So we have a static problem with five equations [3.23 a...e] and five unknowns values of l and c for the young and the old and the capital of the steady state economy, which is equal to savings of the young multiplied by the steady-state interest rate. Again we assumed an interior solution, thus $l_{t,s} < 1$ for all time periods and for both generations alive. This kind of problem can be easily solved by Newton's method for example. If there were disaggregation of the goods and the generations lived for more periods, the bounded or the multiple shooting method might be suitable.

3.4.4 The transition path

For the transition path all generations maximise their utility at the beginning of the first period, except for the old generation alive in the period when the technological change takes place. They have to remaximise their consumption and leisure for their last period. Note that the only optimising decision which the old generation can make, is to select the optimal ratio of consumption and leisure during the last period of its life. Let us say that the change takes place at the beginning of period 1 and that all the effects have taken place by period T. Thus we assume that the dynamic system of our overlapping generations is at a steady-state after T periods or earlier.

After this assumption the mathematical problem can be characterised as follows. Find the vectors $\{c_{1,young} \ldots c_{T,young}\}$, $\{l_{1,young} \ldots l_{T,young}\}$, $\{l_{1,old} \ldots l_{T,old}\}$ and the scalar $c_{1,old}$ so that (1) the budget constraints are satisfied for generations born from period 0 to T: (2) the optimal choice of intertemporal consumption is satisfied for generations born from period 1 to T: (3) the optimal choice of contemporaneous consumption/leisure ratios is satisfied for the generations born from period 1 to T both for the old and the young: (4) the capital at period T must be equal to the capital of the separate final steady-state calculation: (5) all leisure values are strictly under one. Thus we have 3T+1 unknown variables and 3T+1 equations.

3.5 Technological innovation: policy simulation with our overlapping generations model

3.5.1 Description of the algorithm for the numerical solution

We solved the model's steady states using Newton's method and the transition path by arranging the equations as suggested in the normal Fair-Taylor method and applying the Jacobi iteration method with a relaxation parameter. We constructed the transition path algorithm in the following way:

1: We set T=25, the relaxation parameter $\gamma=0.01$ and we assumed an interior solution with respect to leisure, which implies $\mu_t=0$.

2: We guessed the values $K_2 \ldots K_T$, $l_{1,young} \ldots l_{T,young}$ and $l_{1.old} \ldots l_{T.old}$.

3: We calculated the values of L, w and r for all periods and generations.

4: We calculated $c_{t,young}$ and $c_{1,old}$ from capital accumulation equation [3.22]:

$$K_{t+1} = (1 + r_t) * ((1 - l_{t,young}) * e_{young} * w_t - c_{t,young})$$

5: We calculated $c_{t,old}$, with the exception of $c_{1,old}$, from Euler equation [3.20]:

$$\frac{C_{t,old}}{C_{t-1,young}} = \left(\frac{1 + \alpha_0^{\rho} (w^*_{t,old})^{-\rho+1}}{1 + \alpha_0^{\rho} (w^*_{t-1,young})^{-\rho+1}} \right)^{\frac{\rho-\alpha}{\rho-1}} \left(\frac{(1 + r_t)}{(1 + \delta)} \right)^{\alpha} .$$

6: We used equations of the optimal ratio between contemporaneous leisure and consumption and the budget constraint for the new values of K and l. Thus the set of equations we solve by applying the Jacobi iteration method is:

[3.23 a,b,c]

$$l^{i+1}_{t,young} = \left(\frac{\alpha_0}{w^*_{t.young}(K^i_t, l^i_{t,young}, l^i_{t,old})} \right)^{\rho} C_{t,young}(K^i_t, K^i_{t+1}, l^i_{t,young}, l^i_{t,old})$$

$$l^{i+1}_{t,old} = \left(\frac{\alpha_0}{w^*_{t.old}(K^i_t, l^i_{t,young}, l^i_{t,old})} \right)^{\rho} C_{t,old}(K^i_t, K^i_{t-1}, l^i_{t,young}, l^i_{t,old}, l^i_{t-1,young}, l^i_{t-1,old})$$

$$K^{i+1}_t = C_{t,old}(K^i_t, K^i_{t-1}, l^i_{t,young}, l^i_{t,old}, l^i_{t-1,young}, l^i_{t-1,old}) - w^*_{t.old}(K^i_t, l^i_{t,young}, l^i_{t,old})(1 - l^i_{t,old})$$

7: We used the relaxation parameter and went back to step 3 until we achieved convergence.

We chose final steady-state values for the initial guesses and had a constant relaxation parameter (0.01). This led to convergence, but many iteration rounds were needed to calculate the transition path between the steady-states.

3.5.2 Simulation results of a technological innovation in the overlapping generation model.

The technological innovation we simulated with the overlapping generation model was similar to the long-lived agents model, except that the scaling constant ε_1 was ten times as high. Thus the exogenous parameters are from Kenc et al. (1994).

Table 3.2: The parameters of the overlapping generations model.

Parameters of the household	Values	Parameters of the firms	Values (before technological innovation)	Values (after technological innovation)
α_0	4	ε	0	0
α	1	ε_0	1	1
ρ	1	ε_1	10	12
δ	0			

The technological innovation increases utility for all the generations. The increase is naturally the smallest for the generation born at period zero, because the change affects it only in its last period and it was not even able to prepare itself for the change. Therefore there was no perfect foresight in that generation's savings decision.

The interesting feature of the development in utility is the fact that the generation born in period 2 has higher intertemporal utility than its successors and predecessors. The reason the higher utility compared to the successors is the higher interest rate when it is young. The generation born in period two can save almost the same amount with less work than the generation born in period 3, due to the higher interest rate. The firm is still demanding more capital in period two and it is willing to pay for it. The generation born in period 1 has a higher lifetime utility compared not only to its predecessors but also to all of its successors. There are two reasons for this. It has the highest interest rate for its savings and its predecessors have to work "harder", because the lower technology did not allow them to create enough savings for their consumption in period one, leaving more leisure for the generation born in period one.

History dictates lower interest rates for the generation born in period two, which will not be fully compensated by a higher wage rate in that period. The intertemporal utility of each generation is shown in figure 3.4. The transition path of the interest rate is similar to that in the long-lived agents model. The total labour supply first decreases but then temporarily increases in period 2 as figure 3.5 shows. The reason for this is that the generation born in period 1 prefers to transfer its labour to period 2 and enjoy higher wages, as seen in figure 3.6.

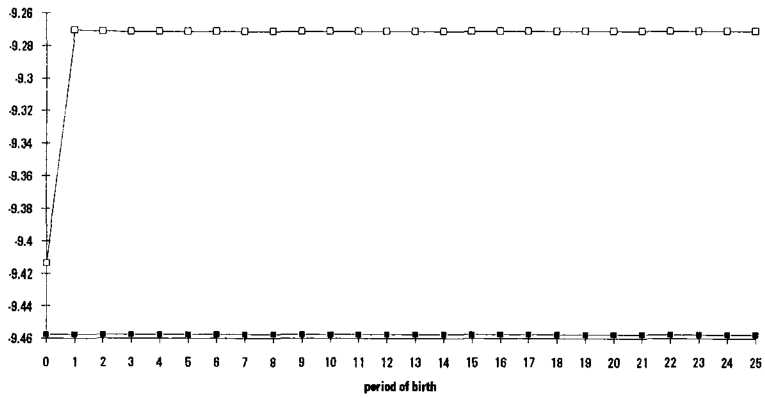

Figure 3.4: The values of the intertemporal utility of each generation after the technological innovation compared to the situation without the technological change (straight line below).

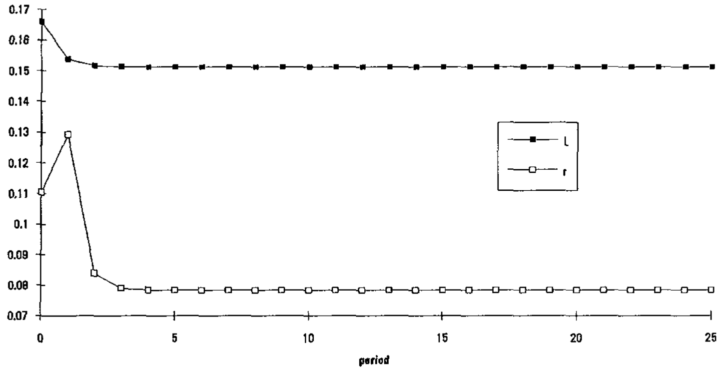

Figure 3.5: Transition paths of the labour supply and the interest rate.

Figure 3.6 shows the accumulation of capital, which is quite similar to the accumulation of capital in the long-lived agents model. Figure 3.6 also shows that leisure for the young generation alive "overshoots" because in periods 2 and 3 the interest rate is higher and the wage rate is lower than in the equilibrium level of the final steady state.

We can see from the figure 3.7 that in addition to leisure, consumption is also higher for the old than the young, which might seem a little odd considering taking into account the positive discount rate. The reason is that the interest rate is higher than the discount rate. Figure 3.8 shows how the goods produced in each period are divided between consumption and additional investment (the area between the two curves).

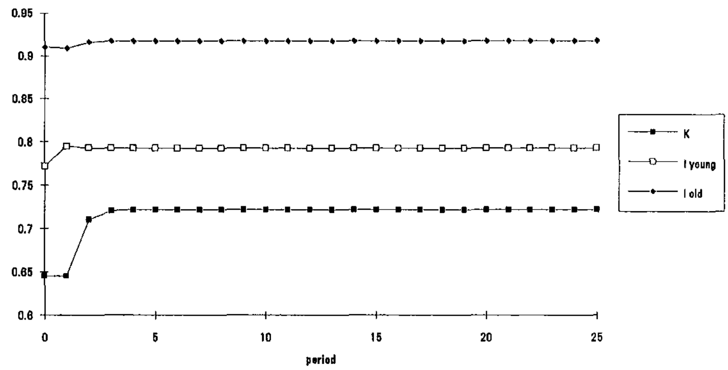

Figure 3.6: The transition paths of capital accumulation and leisure of the old generation alive and the young generation alive in each period.

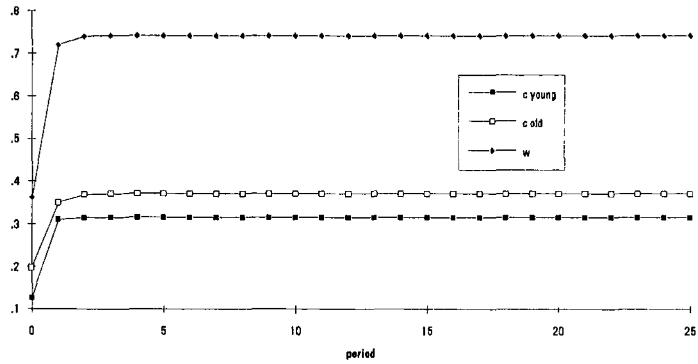

Figure 3.7: The transition paths of the wage rate (divided by 10 for presentation purposes) and consumption of the old generation alive and the young generation alive in each period.

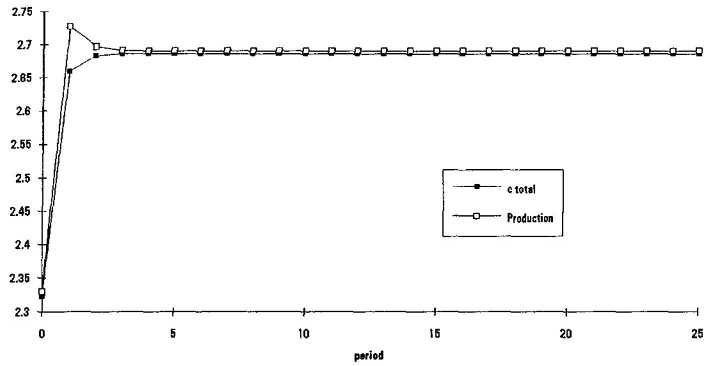

Figure 3.8: The transition paths of total consumption compared to production during each period. The area between total consumption and production is the additional investment in production (more than the previous generation).

3.6 Relative merits of the long-lived agents model and the overlapping generations model.

The formulation of human life and human behaviour is much more realistic in models with overlapping generations structures. The long-lived agents model seems to be used mainly in economic text books. However, we believe that the long-lived agents model structures has its place in policy simulations, which concentrate on production issues and where households are seen only as a source of labour and capital supply, for example, the growth theory [Barro & Sala-i-Martin (1994)]. But when concentrating on issues such as utility distributions between generations, for example, generational accounting, the overlapping generations structure is superior to the long-lived agents model.

4 UNIQUENESS AND STABILITY OF THE DYNAMIC GENERAL EQUILIBRIUM MODELS

In this chapter we discuss several methods for analysing the conditions of stability, the uniqueness of the transition path and possible bifurcations of the solutions to the general dynamic equilibrium models. We also review the comparative statics analysis, which is an additional outcome of the bifurcation theorem. At the end of the chapter we perform a sensitivity analysis on our overlapping generations model described in the previous chapter.

The methods we review and use in this chapter are divided into four parts. Firstly we review the theory of stability and the uniqueness of the transition path to the final steady state. Secondly we apply this theory to our overlapping generations model to check whether we have a unique saddle-point solution. Thirdly we review the theories of bifurcation, comparative statics and the sensitivity analysis. Finally we apply the sensitivity analysis to our overlapping generations model and discuss what it reveals about the stability of the overlapping generations model.

Because of non-linearity we restrict our reviews and analyses to the vicinity of the final steady state, which was calculated for our overlapping generations model in the previous chapter.

4.1 A Method for analysing saddle-point stability around the final steady state

Our analysis for determining whether there is a unique transition path to the final steady state is based on assessing whether the dynamic system is saddle-point stable around the final steady state. Consider a two dimensional space: consumption by the young ($c_{t,young}$) and the capital (k_t).

If the system

$$[c_{t+1,y}, k_{t+1}] = F(c_{t,y}, k_t)$$

is a saddle-point around the final steady state, the transition path is unique. If the system is a source and non-stable, generally there will be no convergence to the final steady state. If the system is a sink (stable) there is a continuum of consumption possibilities for the young generation that maximises their lifetime utility with given predetermined capital and still the economy would converge to the same final steady state [Laitner (1984) and (1990)].

Figure 4.1 illustrates the properties of a saddle point . Let us suppose that the economy is at the initial steady state with the capital at the level where the consumption axis crosses capital axis. Then the technological innovation takes place. The level of capital (state variable) is determined by history, but consumption (control variable) is not determined by history. In the case of a saddle point there is one and only one level of consumption (point a) that leads to the final steady-state. If at that moment a different level of consumption is chosen, for example, point b or point d, the economy does not attain the steady state point (SSP). If the system is a sink there is a continuum of points that lead to the steady state point (SSP), i.e., all points in between point b and point d.

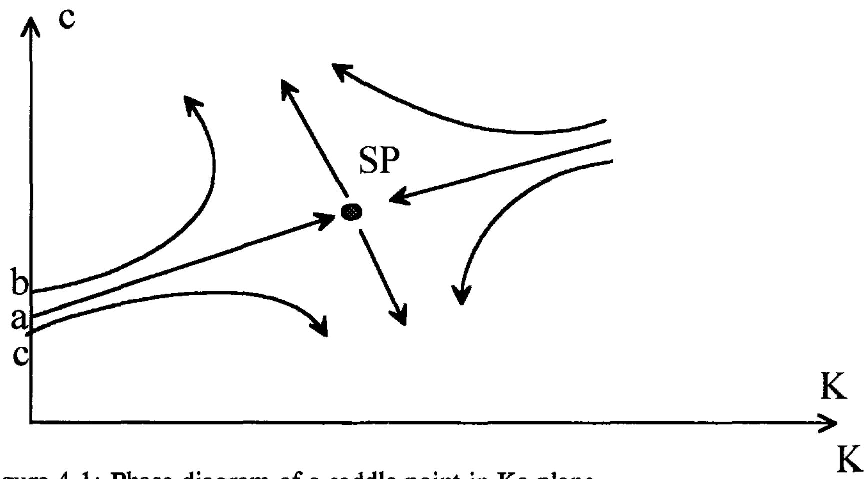

Figure 4.1: Phase diagram of a saddle point in Kc-plane.

A frequently used method for examining the stability properties of a discrete non-linear system is linearization around some point, usually around the fixed point of the system. The disadvantage of the method is that the results are valid only in the neighbourhood of the point of linearization. The radius of the neighbourhood is a function of the error term in the first-order Taylor approximation of the system. An alternative method, where approximations are not needed is, Liapunov's method. But the construction of Liapunov's function for dynamic systems such as the over-lapping generations model is very difficult. For that reason we use the linearization method, which is briefly described here.

Consider a discrete dynamic system, similar to equation system [2.7]:

[4.1] $$x_{t+1} = F\,(x_t, p)$$

where $x_t=[k_t,c_{t,y}]'$. Note that in equation [4.1] there is a relation $c_{t+1,y}=F(k_t,c_{t,y})$, which is different from the optimal relation of the consumption of the old generation at period t+1 and the consumption of the young generation at period t, $c_{t+1,o}=F(k_t,c_{t,y})$. We developed this optimal relation in the previous chapter. In principle we can use the same method to obtain the optimal relation $c_{t+1,y}=F(k_t,c_{t,y})$. We could write the first-order conditions for the young at period t and t+1 and divide them side by side as we have done in the previous chapters. But because both generations have their own budget constraints and a separate Lagrange multiplier associated with it, this leads to two additional variables and the relation is not useful. It is also very difficult or even impossible to study the stability of the steady state of a system, where every generation makes the own dimension in addition to capital.

The vector p represents all the exogenous parameters of the model, for example, the elasticity of substitution in the household's utility function. In this subsection we keep the p vector constant. In the bifurcation and the sensitivity subsections we analyse how the error in the exogenous parameters of the p vector effect the final steady state.

The stability properties of non-linear systems can be analysed in the neighbourhood of the steady-state x* ($x^*_{t+1} = f(x^*_t)$) by using Taylor approximation.

[4.2] $$x_{t+1} = x_t^* + \frac{\partial F}{\partial x}\Big|_{x^*}(x_t - x_t^*)$$

where the Jacobian is

$$\frac{\partial F}{\partial x}\Big|_{x^*} = \left(\frac{\partial x_{t+1}}{\partial x_t}\right) .$$

According to the Hartman-Grobman theorem [Azariadis (1993)] equation [4.2] is topologically equivalent to [4.1] up to a certain distance around the fixed point x*. This allows us to analyse the stability properties of equation [4.1] by examining the eigenvalues of the Jacobian in equation [4.2].

For a system such as our overlapping generations model in the previous chapter we can simply copy the dynamic equation for capital accumulation, but the consumption of the young at period t+1 cannot be solved in the closed form as a function of the consumption of the young at period t and capital at period t, namely in the form $c_{t+1,y} = F(c_{t,y}, k_t, p)$. However, we can write these relations in the form

[4.3] $$G(k_{t+1}, c_{t+1}, k_t, c_{t,y}, p) = 0$$

where G is a system of two non-linear functions:

$$G_1(k_{t+1}, k_t, c_{t,y}, p) = 0, \text{ and}$$

$$G_2(c_{t+1,y}, c_{t,y}, k_t, p) = 0 .$$

We found a steady-state solution to equation [4.3] in last chapter so we have a stationary solution for equation [4.3].

$$G(k_{t+1}, c_{t+1}, k_t, c_{t,y}, p) = 0,$$

where $x_{t+1} = x_t$.

Because the right-hand side of equation [4.3] is zero, we can use the implicit function theorem[15] [Rudin (1989)]:

[4.4] $$\frac{\partial F(x_t)}{\partial x_t} = \frac{\partial x_{t+1}}{\partial x_t} = -\left[\frac{\partial G_t}{\partial x_{t+1}}\right]^{-1}\left[\frac{\partial G_t}{\partial x_t}\right].$$

By analysing the eigenvalues of equation [4.4] we can check whether we have a saddle point [Azariadis (1993)] and [Laitner (1990) and (1984)].

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15 If the inverse matrix in [4.4] exists.

4.2 A Method for analysing saddle point stability around the final steady state: application to the overlapping generations model

In our overlapping generations model the component G_1 in [4.3] is:

[4.5]

$$K_{t+1} - ((1 + r_t(K_t, L_t(c_{t,young}, K_t)) * (1 - l_{t,young}(c_{t,young}, K_t))*$$

$$w_t(c_{t,young}, K_t) * e_{young} - c_{t,young}) = 0$$

We obtain the component G_2 in equation [4.3] by combining [3.21] and the fact that the portion of the good produced but not consumed by the households will be the new invested capital, which is

[4.6] $$K_{t+1} = F(K_t, L_t) - \sum_{s=young}^{old} C_{t,s} + K_t \ .$$

Component G_2 becomes

[4.7]

$$(1 + r_{t+1}(K_{t+1}, L_{t+1}(c_{t+1,young}, c_{t,young}, K_{t+1}))) * (e_{young}*$$

$$(1 - l_{t+1,young}(c_{t+1,young}, K_{t+1}))w_{t+1}(c_{t+1,young}, K_{t+1}) - c_{t+1,young})$$

$$+ \sum_{s=young}^{old} c_{t+1,s} - (1 + r*_t (K*_t, L_t(c*_{t,young}, K*_t))) * (e_{young}*$$

$$(1 - l*_{t,young}(c*_{t,young}, K*_t))w_{*t}(c*_{t,young}, K*_t) - c*_{t,young}) = 0$$

The deep implicit structure of both G_1 and especially G_2 makes the use of the analytical solution method of section 4.1 quite difficult to apply. Instead we apply direct numerical derivation around the final steady state.

We solve the differentials of equation [4.4]

$$-\left[\frac{\partial G_t}{\partial x_{t+1}}\right]^{-1} \quad \text{and} \quad \left[\frac{\partial G_t}{\partial x_t}\right]$$

numerically around the final steady state. The function G is defined by the equations [4.5] and [4.7] and the values of the exogenous parameters in equations [4.5] and [4.7] are the values after the technological change: table 3.2. We perform the numerical calculation by calculating G_1 and G_2 at the final steady-state. The values are practically zero, as they should be. Then we add a derivation step (e=1e-10) to each of the variables K_t, $c_{t,y}$ $c_{t+1,y}$ and k_{t+1} one by one and subtract the new G values from the G values before the derivation step and divide it by the derivation step, namely:

$$\frac{G(x+\varepsilon)-G(x)}{\varepsilon}$$

The right-hand side of equation [4.4] becomes

$$\frac{\partial F(x_t)}{\partial x_t} = \begin{pmatrix} -1.07 & 0.68 \\ 1.55 & -0.91 \end{pmatrix},$$

and the eigenvalues are 0.0436 and -2.02, which implies [Azariadis (1993)] that we have a saddle point around the final steady state. The solution of the final steady-state is very likely to be unique, at least the transition path in the neighbourhood of the final steady-state is unique.

4.3 Bifurcations of equilibrium, comparative statistics and the sensitivity analysis

To define a bifurcation point of equilibrium, consider the steady-state version of equation [4.3]:

[4.8] $$G(x^*,p)=0$$

When the exogenous parameters (p vector) vary, so will the fixed point of the system. In most cases small changes in p will not affect the qualitative structure of the orbits of the dynamic system. For some critical values of p, however, small perturbations can lead to qualitative changes in the system's orbit structure and its dynamic behaviour. When this happens, we say that a bifurcation has occurred [Azariadis (1993)].

The mathematical methods for analysing possible bifurcation are quite similar to those for analysing the uniqueness of the transition path, because both methods use the implicit function theorem. We can write equation [4.8] in the implicit form :

[4.9] $$G(x^*(p),p)=0$$

and utilising the implicit function theorem we obtain a useful result for comparative statistics analyses of the final steady state

[4.10] $$\frac{\partial x^*}{\partial p}=-\left[\frac{\partial G}{\partial x^*}\right]^{-1}\left[\frac{\partial G}{\partial p}\right].$$

This comparative statistics results tell us in which direction the position of the final steady state changes following the shock.

The bifurcation point is where $\frac{\partial G_t}{\partial x*}$ is not a full rank, i.e. if its determinant is zero.

The disadvantage of this method is the fact that the p vector includes several parameters which should be estimated from available data on the economy we want to study, and therefore all the parameters are entitled to have some error simultaneously. Therefore a widely used[16] alternative is sensitivity analysis.

In sensitivity analysis the same model is solved again with slightly modified parameters and checked to see to what extent the final steady state and the points in the transition path vary. This is then repeated with different modification of the exogenous parameters.

All the exogenous parameters are entitled to have at least some simultaneous error, because they are estimated from the historical data. Therefore sensitivity analysis should always be made in addition. The sensitivity analysis should include positive or negative variation of all the exogenous parameters at the same time [Auerbach & Kotlikoff (1987)].

4.4 Sensitivity analysis of the overlapping generations model

We performed sensitivity analysis to determine how a changes in the exogenous parameters alter the final steady state. We added one percent to each of the exogenous parameters one by one and calculated the change in the final steady-state. In our final sensitivity analysis we added one percent to each of the exogenous parameters simultaneously. The results are given in the table 4.1 and an illustration of the change in the control and the state variable can be seen in figure 4.2. The values in the table 4.1 are percentage changes in the steady state values of the endogenous variables after a change in the exogenous parameters. The final row shows the

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16 For example Auerbach & Kotlikoff (1987) and Perraudin & Pujol (1991)

change in the endogenous variables after increasing all the exogenous parameters by one percent simultaneously.

Table 4.1: Percentage change in the steady state values of the endogenous variables after 1% change in the exogenous parameters.

	K	l young	l old	L	c young	c old	c total	production	w	r
$\varepsilon 1$	0.3666	0.116	0.023	-0.3973	0.6685	0.575	0.6208	0.6146	1.0641	-1.9778
$\varepsilon 0$	0.7359	0.0585	0.0559	-0.3214	0.466	0.4634	0.4647	-0.2996	0.0872	-4.0487
ε	1.0266	-0.0008	0.0828	-0.2238	0.2367	0.3205	0.2795	0.2858	0.4573	1.7299
$\alpha 0$	-0.2539	0.0936	0.0612	-0.4371	-0.415	-0.4473	-0.4315	-0.4333	0.0154	-0.7174
α	0.3674	-0.0309	0.0307	0.0051	-0.0152	0.0464	0.0163	0.0127	0.0302	-1.4061
σ	-0.034	0.0047	0.0164	-0.0585	-0.0105	-0.1031	-0.0577	-0.058	0.0021	-0.096
ρ	0.2232	-0.0901	-0.0455	0.3843	0.4099	0.3499	0.3793	0.3809	-0.0135	0.6308
e0	2.458	-0.0567	0.2056	0.3573	0.55	0.2935	0.4191	0.3996	0.1689	-7.797
e1	-1.598	0.1369	-0.1397	-0.0136	0.054	0.2952	0.1771	-0.0479	-0.1372	6.4515
all	3.197	0.2382	0.2553	-0.3982	1.9963	1.9073	1.9509	1.9464	2.3743	-2.1265

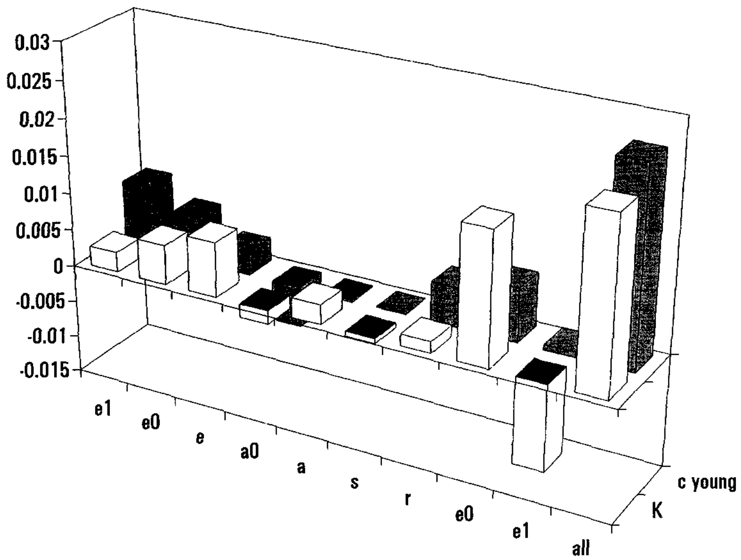

Figure 4.2: The changes in the control and state variables caused by a small shock in the exogenous parameters.

The change in the exogenous parameter e_{young} changes the amount of the production capital most. The parameter e_{young} is the work efficiency of the young and it is logical that the production capital, which is equal to the savings of the young, reacts strongly to e_{young}. The change in e_{old} also causes an effect of almost the same size in terms of absolute values. The negative effect on the production capital by an increase in e_{old} is very logical. If the households can earn more when they are old without wasting any more of their leisure time, they will not want to save up so much for their old days, but consume or have more leisure when they are young, before the discounting term "bites". The changes caused by the other exogenous variables are significantly smaller in terms of absolute values.

The change in the production capital caused by small changes in the exogenous parameters is relatively small. Thus we can say that small variations in the exogenous parameters will not cause huge changes in the amount of the production capital.

In terms of consumption when young, the change in the parameter e_{young} also has high absolute value. The change in the parameter ε, the scaling constant for production, has the highest absolute value. A small positive change in both of these exogenous parameters causes a positive change in consumption when young. This is logical, because both these parameters offer the young more possibilities to consume, without forcing them to give up any leisure at any period or any consumption when old. The change in the exogenous parameter α_0 has the greatest negative effect on consumption when young. This is also logical because an increase in the leisure preference parameter α_0 means that the individual prefers more contemporaneous leisure than contemporaneous consumption.

The one percentage simultaneous changes in the exogenous parameters cause the biggest change in the production capital, which is a little over three percent. This is not a sufficiently significant change, to be worried about. The sign of the change in each endogenous variable caused by the change in each of the exogenous parameters is also logical.

Overall sensitivity analysis allows us to assume that a small variation in the exogenous parameters will not cause significant changes in the position of the final steady-state.

5 CONCLUSIONS, CRITICISM AND FURTHER APPLICATIONS

In this final chapter we draw conclusions from the work that we have done. We also discuss what kind of dynamic general equilibrium model is suitable for modelling economic reactions for generational accounting. What kind of numerical methods are practical for this particular model and how should its stability properties be analysed? At the end of the chapter we review critically the weaknesses of the dynamic general equilibrium models as an economic method for analysing outcomes of the reactions of economic agents in various circumstances. We also discuss very briefly how to overcome or live with these weaknesses.

5.1 Conclusion

As stated in the introductory chapter we have divided the main purpose of this study into three themes.

The first theme was to study different numerical methods for solving discrete dynamic general equilibrium models. We solved the optimal growth model using several numerical methods. We also solved a simple policy simulation using two self designed models.

Our main findings are: the multiple and bounded shooting methods are very powerful numerical methods, especially for discrete dynamic equilibrium models. The dynamic structure of the overlapping generations models makes them difficult to apply. The bounded and multiple shooting methods are very suitable for the general dynamic equilibrium models, which have no overlapping generations structure or the steady-state calculations of the models with the overlapping generation structure. For the transition paths of the overlapping generations model we recommend arranging the equations as suggested in the Fair-Taylor method and then applying either Jacobi or Gauss-Seidel fixed point iteration methods as here and in Auerbach & Kotlikoff (1987) or using Newton's search as in Perraudin & Pujol (1991).

The second theme was to construct dynamic general equilibrium models and study how the conditions of optimality are obtained under the assumption of perfect foresight. We constructed two models, one with the long-lived agents structure and one with the overlapping generations structure. These models are based on the models in Auerbach & Kotlikoff (1987), Perraudin & Pujol (1991), Catez et al. (1992) and Söderling (1989). We derived the conditions under which rational households and firms behave under perfect foresight in both of these models. We did this using both recursive discrete dynamic optimisation and the Kuhn-Tucker conditions of optimality. We have also simulated the consequences of a technological innovation. We did this by solving both models numerically before and after a surprise shock in the production function.

The third theme was to review some of the stability theories related to the discrete general dynamic equilibrium models. We reviewed theories for determining the local uniqueness of the transition path, bifurcations of the equilibrium, the comparative statics of the final steady state and the sensitivity analysis. We also applied the theory of local uniqueness of the transition path and sensitivity analysis to our overlapping generations model. Our main findings concerning the uniqueness of the transition path and bifurcation theorem was that these theories are very difficult to apply and they do not give much extra information. Therefore such calculations are not usually executed by designers of the dynamic general equilibrium models, excluding Laitner (1984) and (1990). Other designers of large-scale policy simulation models have contented themselves with sensitivity analysis and its results. This is perhaps because the other methods require a lot of work and seldom give much additional information compared to the sensitivity analysis. This brings up a question, when one is analysing economics with difference equations and when one is only analysing difference equations?

5.2 A proposal for expanding the generational accounting approach with a dynamic general equilibrium model

The key issue in generational accounting models is the comparison between different generations' payments to the government and social benefits received from the government. Therefore the dynamic general equilibrium model should have an overlapping generations structure, where each generation lives, for say, five periods. Labour efficiency should be set at zero for the final period of each generation, implying retirement in that period.

Secondly the government should be modelled. The government itself does not maximise anything; it just maintains the long term budget balance. This requirement is provided by the following equation, which implies that the current value of the future taxes must be equal to the current value of the future government spending minus the initial net government debt.

[5.1] $$\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}(1+r_s)\right)T_t=\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}(1+r_s)\right)G_t+D_0,$$

where T_t is the taxes, collected in period t and G_t is government spending on social and other services in period t. D_0 is net government debt at the initial steady state and r_t is the interest rate. We should let the tax variable T include three types of taxes, namely, income taxes, sales taxes and capital taxes. This does not change the behaviour of the firm (we keep the tax free price of the good as a numeraire) or the direct utility function of the household. It changes the households' budget constraint in our overlapping generations model together with the fact that generations live for more periods, thus.

[5.2] $$\sum_{t=1}^{T}\frac{e_{age}w_t(1-\tau_L)*(1-l_t)-(1+\tau_c)c_t}{\prod_{s=2}^{t}(1+r_s(1-\tau_K))}=0\ ,$$

or as a form of discrete state equation:

[5.3] $$a_{t+1} = (1 + r_s(1 - \tau_K)) * a_t + e_{age}w_t(1 - \tau_L) * (1 - l_t) - (1 + \tau_c)c_t \ ,$$

where a_0 and a_T are zero and where t_L, t_c and t_K are the tax rates for wages (income tax), consumption (sales tax) and capital tax respectively. The other variables are defined in chapter 3. Because we force each generation to retire in its final period we cannot assume an interior solution in respect of the leisure constraint in all periods of the households' life.

In our opinion, the best way to solve this model is to apply Newton's method for the steady states. For the transition path we suggest arranging the equation as suggested by Fair & Taylor (1983) and apply the Jacobi iteration. Because we cannot assume an interior solution we must calculate leisure from the contemporaneous consumption leisure ratio. If leisure becomes over 100% of all time in a period, we would have to set the Lagrange multiplier associated with that particular constraint so that the contemporaneous consumption leisure ratio formula yields leisure value as 100% of all time in that period, which is in line with the Kuhn-Tucker theorem. For analysing the stability properties of this model, sensitivity analysis should be adequate.

5.3 Some weaknesses of the dynamic general equilibrium approach and suggestions for overcoming these weaknesses

The use of numeric dynamical general equilibrium models by economists has increased in tandem with computer technology. It is important to bear in mind the limitations of dynamic general equilibrium models one uses. Here we list some of the limitations of our models and discuss how these limitations should be taken into account when performing policy simulations.

The main weakness is the estimated exogenous parameters. In all the models we have read for this study, the exogenous parameters of, say, the household's utility functions are constants over time. So when comparing the utilities of different generations with a model which has an overlapping generations structure, we must state that different generations in the model have the same values in their lives. On the other hand we do not know what value future generations will attach to different things, so it seems reasonable to use the utility function estimated for the current generations for future generations also.

The assumption of perfect foresight has been used in almost all dynamic general equilibrium models world-wide without there being any analysis how it affects the position of the solution. Our opinion is that the assumption of perfect foresight is an oversimplification. Azadiaris (1993), however, has introduced an alternative method, referred to as least squares learning. Under the assumption of perfect foresight, the economic agents are able to calculate future prices with certainty in non-stochastic model structures. In least squares learning, agents forecast the future prices using an autoreggressive formula, which has historical prices and their derivatives as dependent variables. It is our hope that this or a superior method will oust the assumption of perfect foresight from dynamic general equilibrium models in the near future.

We can argue that the unemployed person is unemployed in Finland, say, because he refuses to take a job that pays FIM 1 per hour. Hence his unemployment is voluntary. But if the dynamic general equilibrium model is used in reality to analyse employment issues, it should have a better way of modelling unemployment, simply for the sake of the its plausibility. One way to do this is to introduce trade unions into the model as in Jensen et al. (1993). In this concept the workers authorise the trade unions to negotiate the wage rate and the labour supply with the employees.

It is often stated that by choosing the right parameters we can obtain whatever results we desire. That is quite true, and therefore it must be stated clearly how parameters are estimated and sensitivity analysis should be performed. An interesting idea would be a write a program, which generates different parameter values for sensitivity analysis by minimising the absolute value of the determinant of $\frac{\partial G}{\partial x*}$ in equation [4.10]. Thus in addition to performing sensitivity analysis the program search for bifurcation points.

A lot of aggregations also have to be made when constructing dynamic general equilibrium models. In dynamic models all the households in the economy must be aggregated to just a few households. And the same is true for the firms in the economy. This makes it impossible to study how different policies affect different types of households and firms. In theory, of course, it is possible to add as many different types of households and firms as one wants, but the problems of solving the models increase vastly and thus this has not been done in practice. These types of studies usually use a static equilibrium approach. In models with an overlapping generations structure the households of different generations have been disaggregated, and therefore comparisons between generations can be made.

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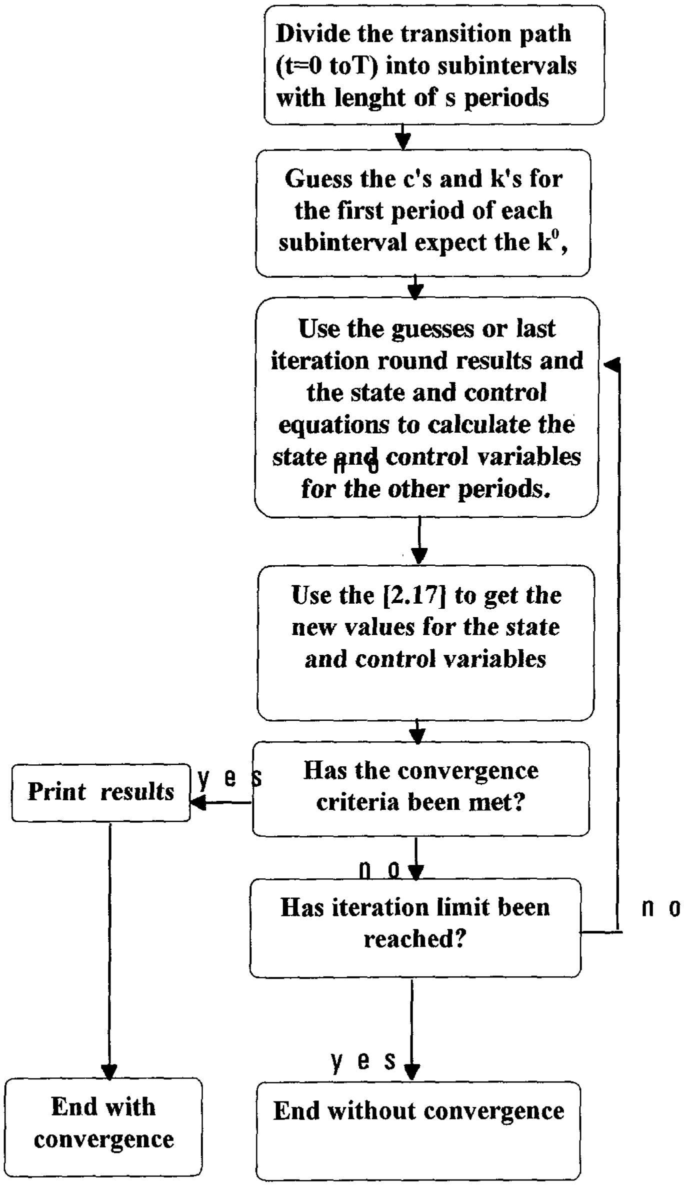
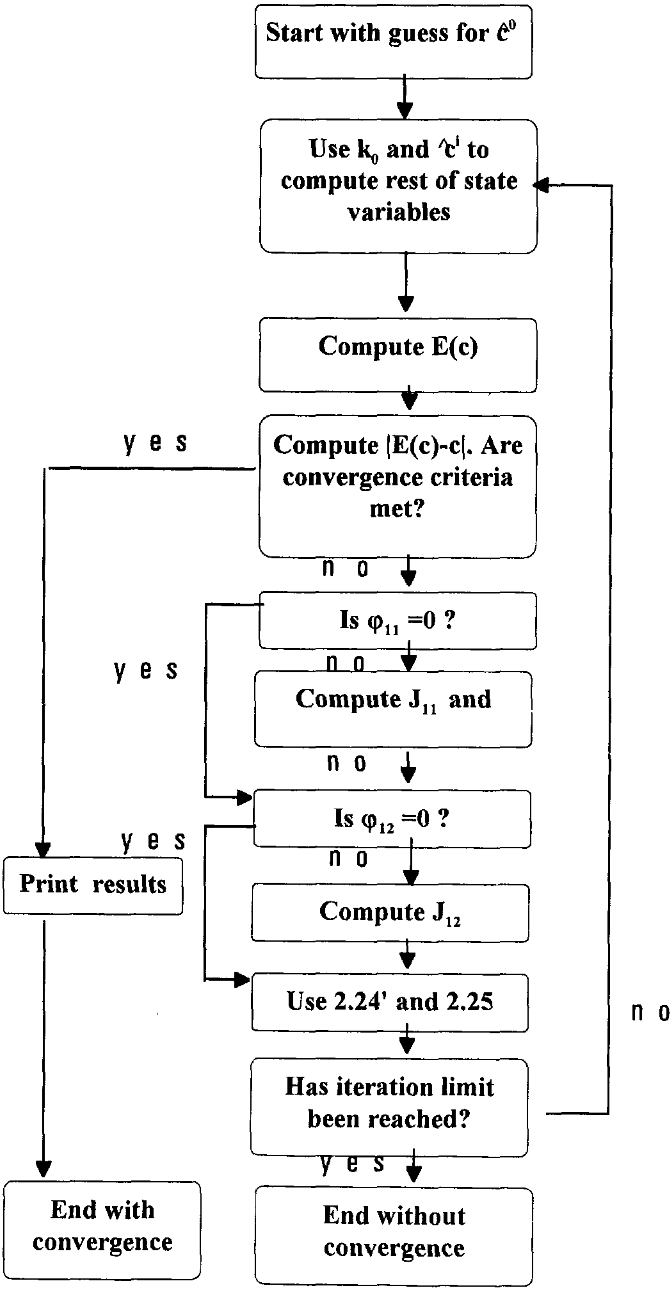
Appendix 1

FLOW DIAGRAM FOR THE MULTIPLE SHOOTING METHOD
- A DISCRETE CASE



FLOW DIAGRAM FOR WILCOXEN'S MODIFIED FAIR-TAYLOR METHOD



Appendix 2

Generational accounting

Generational accounting is based on the fact that the present value of the future net tax payments of the current and future generations must be sufficient to cover the present value of future public consumption, as well as pay the initial public indebtedness. Failure to satisfy this constraint means that the government will default on its liabilities, in essence satisfying the constraint through a tax on its creditors. If it does not mean that the public debt will be paid off, the constraint is satisfied. The only requirement is that the debt is serviced through tax payments by existing and future generations. This constraint can be expressed by a simple equation:

$$EGT+FGT=FGC-GNT,$$

where EGT is the present value of the remaining net tax payments of existing generations, FGT is the present value of the net tax payments of future generations, FGC is the present value of all future public consumption and GNT is public net wealth.

In the generational accounting approach EGT and FGT are dependent on the labour supply, savings rate, wage rate etc., which are naturally projections. In the basic generational accounting approach these projections are considered to be independent of projections of FGC.

This study analyses dynamic general equilibrium models from a mathematical point of view, and in addition it attempts to construct a dynamic general equilibrium model, suitable for expanding the generational accounting approach so that projections of the labour supply, saving rate, wage rate etc. are functions of the projections of FGC. [Auerbach et al. (1994)].

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