SPECIMEN MATHEMATICUM,

De

Methodo Superficies Solidorum duplici integratione investigandi.

Venia Ampl. Fac. Phil. Ab.

ad publicum examen deferunt

AUCTOR

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&

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Horis antemeridianis.

ABOÆ,

Typis FRENCKELLIANIS.

VIRO

SPECTATISSIMO ET CONSULTISSIMO,

URBIS TAVASTEBURGENSIS NUPER CONSULI

DEXTERRIMO,

D:NO *GOTTLIEB JOHN*,

REGG. SOCIETT. PATR. SVECICÆ ET OECON. FENNICÆ

MEMBRO,

FAUTORI PROPENSISSIMO.

Ob beneficia in se abunde collata

D. D. D.

AUCTOR & RESPONDENS.

VIRO

ADMODUM REVERENDO ATQUE DOCTISSIMO,

D:NO *GABRIELI PALANDER*,

ECCLESIÆ IN WÅNÅ PASTORI MERITISSIMO,

PARENTI INDULGENTISSIMO.

Pietatis in Nomen Paternum colendæ studio, ut nihil esse sanctius antiquiusve mortalium sciscunt sapientissimi; ita nihil quoque esse jucundius & ad humanam sortem beandam adcommodatius, bene nata sentiunt pectora, imo olim sentire pergent. Tanti igitur officii svavissima excitatus conscientia, hanc Tibi, Genitor Optime, dissertatiunculam, Tuis pene infinitis, de mea felicitate fovenda, curis, laboribus atque impensis omnino debitam, offerre gestio: leve sic quoddam tenerrimæ, qua animus calet, pietatis monumentum positurus. Quod ut Paterna adspicias fronte, supplex obsecro; dum in vivis ero, futurus Tibi,

PARENS INDULGENTISSIME,

filius obsequentissimus
GABRIEL PALANDER.

§. I.

Doctrinam Solidorum, quamvis & eximio insignem usu & quæstionum haud spernendarum copia abundantem, ceteris tamen Geometriæ partibus, sive inventorum molem sive propositorum evidentiam spectes, longe posteriorem remansisse, ut certis constat indiciis, ita peritis harum rerum arbitris mirum videri profecto haud poterit. Enimvero Geometriæ Curvarum excolendæ cupiditas studiumque pene totos diu Geometras detinuit, cum quod ejus in Physicis magis obvium sentirent usum, tum maxime, quod Curvarum auctior penitiorque cognitio, & viam apertura esset promtiorem, & ditiorem ipsis allatura instrumentorum penum, ad abstrusiorem illam Solidorum ac Superficierum naturam felicius demum rimandam. Nempe una erat hodieque est via, reconditam *Solidorum* naturam indagandi, ea nimirum, quæ sectionibus quaquaversum faciendis initur: quæ ipsæ cum *Curvæ* sunt; *harum* prius ut eruerentur affectiones, necesse erat, quam in indole *illorum* exploranda cum fructu versari possent. Sic v. gr. ex Curvarum rectificatione, Superficierum investigationem; ex quadratura, Soliditatis indagationem, pendere omnem quis non videt?

In corporum ſive Soliditatibus ſive Superficiebus inveſtigandis, Analyſin adhibentes infinitorum, hac vulgo via incedunt Mathematici, qua ea, rotatione curvæ circa fixum quemvis axem, genita ſpectantur: quæ conſideratio facilem & egregiam præbet rationem fluxiones Superficierum pariter atque Solidorum inveniendi. Nempe, ex data æquatione Curvæ rotantis, elementum ejusdem quæratur, quod, dein ductum in peripheriam Sectionis Circularis normaliter ad axem factæ, exprimet fluxionem Superficiei integrando demum exhibendæ. Solidi autem fluxio invenitur, aream ipſius ſectionis in fluxionem axeos ducendo: qua rite integrata, Soliditas quæſita prodibit.

Eſt vero & altera a Geometris haud raro adhibita ratio, corpora ſcil. conſiderandi, utpote motu plani parallelo, juxta lineam normaliter ipſi inſiſtentem, quæ *Directricis* venit nomine, exorta. Quæ Methodus, cum in Solidis inveſtigandis ab illa, quam expoſuimus, parum modo differat, Superficiebus autem inveniendis paullo ineptior videatur; eam fuſius repetere jam non vacat.

Methodos hactenus a nobis traditas, etſi inſignem in doctrina Solidorum uſum præſtent, maximis tamen obnoxias eſſe difficultatibus atque defectibus, facile depre-

deprehendet, quicunque accuratiori rem lance pensitaverit. Etenim longe plurima Superficierum sunt genera, de quarum per motum ortu non constat: imo nonnulla, quarum cognita genesis, ad quæstionem, de Superficie invenienda, solvendam, nil conferre opis valeat; quo ex genere habeas *Coni scaleni Superficiem*, quæ allatas Methodos vix admittere videtur.

Quibus supplendis defectibus haud inanem contulisse nobis videtur operam Cel. LEONHARDUS EULERUS in *Diss., de Formulis integralibus duplicatis, Nov. Comment. Acad. Petrop. Tom. XIV. P. I. inserta*, in qua duplicis integrationis, ita scil. instituendæ, ut in priori binarum variabilium una, in posteriori altera sola variabilis habeatur, vim atque in Stereometria usum exponit. Quæ Methodus, cum & longe latius patere, &, quam melius excolant Geometræ, digna nobis videatur, Specimine qualicunque Academico, ejus explicare naturam, atque in Superficiebus, ex lege æquationis Algebraicæ continuis, eruendis ostendere usum constituimus. Tenuitatis autem virium maxime nobis conscii, L. C. exoremus oportet, ut ausis juvenilibus benigniorem adspirent auram.

§. II.

Quo natura Methodi a nobis tradendæ facilius cognoscatur, de affectionibus *Formularum integralium*, quas cum Eulero *duplicatas* appellare placet, nonnulla, e re est, præmittantur. Primo quidem, ne ancipiti signorum interpretationi ullus relinquatur locus, monendum est, formulam nostram geminato integrationis signo affectam, & hac forma: $\int\int Zdxdy$ (existente Z functione ipsarum x & y) conspicuam, ab his: $\int Xdx . \int Udx$ & $\int (Xdx \int Udx)$ (denotantibus X & U functiones unius x) accurate esse distinguendam. Quarum nempe altera productum binorum integralium, altera integrale ipsius Xdx ducti in $\int Udx$ exprimit: utraque duplici quidem integratione eruenda, attamen a formula nostra primum proposita eo discrepans, quod, cum harum binæ integrationes per unam variabilem x peragantur, nostra illa, utpote producto: $dxdy$ affecta, ita tractetur, ut in altera integratione sola x, in altera y variabilis existimetur.

Quo constituto discrimine, formulæ quoque nostræ duplicem inesse vim observare licet, exinde oriundam, quod quantitates binæ variabiles x & y aut nullo prorsus nexu cohæreant, aut aliqua inter se relatione connectantur. Qui uterque casus seorsim est spectandus.

Quod si prior obtineat locum, res eo redit, ut ea quæratur functio ipsarum x & y, quæ ita bis differentiata, ut primum x mox y sola variabilis putetur, hanc formam: $Zdxdy$ exhibeat. Sic formulam: $ax^m y^n dxdy$, fluxionibus, uti innuimus, bis sumendis,

ex

ex hac: $\frac{ax^{m+1}y^{n+1}}{m+1 \cdot n+1}$ exortam facile videmus: unde $\iint ax^m y^n dxdy = \frac{ax^{m+1}y^{n+1}}{m+1 \cdot n+1}$.

Hæc igitur erit regula hujusmodi formulas integrandi, ut investigetur primum integrale $\int Zdx$, solam x spectando ut variabilem, quod erit functio quædam ipsarum x & $y = V$, deinde vero ducatur V in dy; denuoque instituta integratione, eruatur $\int Vdy = \iint Zdxdy$. Quo autem integrale obtineatur completum, functiones arbitrarias X & Y, illam ipsius x, hanc ipsius y, quippe quæ integrando tolluntur, addendas esse, ex ipsa differentiandi regula optime patet.

At longe ab hoc differt alterum illud, quod dixi, formularum integralium duplicatarum genus, in quo variabiles x & y mutua quadam relatione continentur. Nempe priori peracta integratione, in qua v. gr. y sola variabilis habebatur, extremus quisque valor quem ipsa y recipere valuerit, in formula denuo integranda ejus loco erit substituendus; qui cum plerumque sit functio quædam ipsius x, haud parum abest, quin in altera integratione ipsa y constantis vice fungatur. Sic, si fuerit proposita formula: $\iint ax^m y^n dxdy$, definito simul limite variationis x & y hac æquatione: $y^r = x^s$; tum invento primum $\int ax^m y^n dy =$

$= \frac{ax^m y^{n+1}}{n+1}$, extendatur ipsa y usque ad terminum $x^{\frac{s}{r}}$; quo valore substituto, formula: $\int \frac{ax^m y^{n+1}}{n+1} dx$ abit in hanc: $\int \frac{ax^m . x^{\frac{\overline{n+1}.s}{r}} dx}{n+1} = \frac{ax^{\frac{\overline{m+1}.r+\overline{n+1}.s}{r}}}{(m+1).(n+1)}$.

Si ab integranda formula: $\int ax^m y^n dx$ initium fecissemus, obtinendum nobis fuisset $\int\int ax^m y^n dx dy =$ $\frac{ay^{\frac{\overline{m+1}.r+\overline{n+1}.s}{s}}}{(m+1).(n+1)}$.

Quod genus cum totum ad doctrinam Solidorum pertineat; ejus ratio ex sequentibus uberius patebit.

§. III.

PROBLEMA. *Data æquatione solidi inter tres coordinatas x, y & z, elementum superficiei, quod rectangulo infinite parvo dxdy in basi sumto imminet, invenire.*

Constituto *(Fig. 1.) AP* axe abscissarum, ex quovis superficiei puncto *M* demittatur *MQ* perpendicularis in basin planam per axem transeuntem, atque dehinc *QP* normalis in *AP*. Sint: $AP = x$, $PQ = y$, $MQ = z$; quo facto patet, Superficiei naturam ex mutua ha-

harum trium coordinatarum relatione pendere, & idcirco æquatione inter x, y & z exprimendam esse. Unde, cum z sit functio quæpiam duarum variabilium x & y, fluxio ejus æquatione definietur hujus formæ: $dz = Pdx + Qdy$ (P & Q denotantibus functiones ipsarum x & y); quæ æquatio, alterutra variabilium x & y assumta constanti, abit in hanc: $dz = Pdx$, sive hanc: $dz = Qdy$. Quo autem exhiberi queant hæ fluxiones; sumatur $Pp = dx$, $Qq = dy$: ducantur Qs, qd parallelæ Pp, atque pd parallela Pq: erigantur perpendiculares: sh, qn, dl, superficiem in punctis h, n, l attingentes; quo facto sectiones, ex lateribus columellæ elementaris, in Superficie oriundæ erunt Mh, Mn, ln, hl, spatiumque his interceptum erit parallelogrammum. Ducatur porro recta nh & sumatur: $sH = qN = dL = QM = z$; unde erit: $MH = NL = dx$ & $MN = HL = dy$, & diagonalis, quæ ducatur, $NH = \sqrt{dx^2 + dy^2}$. Jam vero duo sponte exstant elementa ipsius z, unum Hh ex fluxione ipsius x dependens, alterum Nn ex parum variata ipsa y enatum. Quorum idcirco illud $= Pdx$, hoc $= Qdy$. Hinc porro habetur:

$$Mh = \sqrt{MH^2 + Hh^2} = \sqrt{dx^2 + P^2dx^2} = dx\sqrt{P^2 + 1}.$$

$$Mn = \sqrt{MN^2 + Nn^2} = \sqrt{dy^2 + Q^2dy^2} = dy\sqrt{Q^2 + 1}.$$

Quibus erutis valoribus eo redactum est opus, ut ex dato rectangulo $NH = Qd = dxdy$, parallelogrammum nh, lateribus $Mh = dx\sqrt{P^2 + 1}$, $Mn = dy\sqrt{Q^2 + 1}$ compreprehensum quæratur. Cujus Problematis solutio cum data diagonali nh facile succedat; valorem ejus, ex cognitis: Hh, Nn, NH, age, primum definiamus:

Si fuerit $Hh = Nn$; mox patet esse $nh = NH = (dx^2 + dy^2)^{\frac{1}{2}}$ ob Hh & Nn parallelas.

Sin minus, altera scil. Nn existente majore, ducatur hO parallela ipsi NH. Unde $nO = Nn - Hh = Qdy - Pdx$ atque $nh = \sqrt{hO^2 + nO^2} = \sqrt{dx^2 + dy^2 + (Qdy - Pdx)^2}$.

Dein vero demittatur ex n no perpendicularis in Mh, quo facto habetur $nh^2 = Mh^2 + Mn^2 - 2Mh . Mo$, (*Elem. Eucl. Libr. II. Prop. XIII.*) atque hinc: $Mo = \frac{Mh^2 + Mn^2 - nh^2}{2Mh}$, $no = \sqrt{Mn^2 - Mo^2} = \frac{\sqrt{2(Mh^2 . Mn^2 + Mh^2 . nh^2 + Mn^2 . nh^2) - Mh^4 - Mn^4 - nh^4}}{2Mh}$

Area denique ipsa parallelogrammi $nh = Mh . no = \frac{1}{2}\sqrt{2(Mh^2 . Mn^2 + Mh^2 . nh^2 + Mn^2 . nh^2) - Mh^4 - Mn^4 - nh^4}$. Quæ æquatio, restitutis valoribus ipsarum Mh, Mn, nh supra inventis, & reductione facta præbet elementum Superficiei rectangulo: $dxdy$ imminens $= dxdy\sqrt{P^2 + Q^2 + 1}$. *)

Schol.

*) Formula hæc ab EULERO suis in scriptis aliquoties adhi-

Schol. Si fuerint elementa ipſius z diverſi nominis (*Fig.* 2.), alterum videlicet Hh poſitivum, alterum Nn negativum; habebitur $nO = Hh + Nn = Pdx + Qdy$ ideoque $nh = \sqrt{dx^2 + dy^2 + (Pdx + Qdy)^2}$; qui valor ſi in formula inventa pro area ipſius parallelogrammi nh ſubſtituatur, reducendo obtinebitur expreſſio: $dxdy\sqrt{P^2 + Q^2 + 1}$, eadem quæ ſupra.

§. IV.

PROBLEMA. *Data æquatione Solidi cujusvis naturam exprimente, Superficiem invenire.*

Sumto, uti §. præcedenti (*Fig.* 1.), plano APQ pro baſi, quæ Superficiem ſecet per Curvam AG, prioribusque retentis deſignandi modis, patet ſitum cujusque Superficiei puncti M coordinatis $AP = x$, $PQ = y$ & $QM = z$ determinari.

Ad inveniendam Superficiem baſi huic reſpondentem, æquatio data differentietur, atque ex ea dein eruatur valor formulæ: $dxdy\sqrt{P^2 + Q^2 + 1}$, cujus integrale $\iint dxdy\sqrt{P^2 + Q^2 + 1}$, iuſtis adhibitis determinationibus Superficiem exhibebit quæſitam.

Duplicem vero formulæ noſtræ integrationem, ut diverſa objicitur quæſtio, ita rationibus non parum diver-

bita occurrit: quam quomodo invenerit, anxie fruſtraque disquirentes, ejus eruendæ periculum ipſi fecimus.

diversis peragendam esse facile monstrabimus. At duas maxime placet enodare quæstiones, alteram de Superficie toti basi *APG* imminente, alteram de ea tantum portione ejus, quæ rectangulum *FP* obtegit, invenienda. In *illa* expedienda ita versandum, ut sumto primum integrali $\int dy \sqrt{P^2 + Q^2 + 1}$, habita x constanti, promoveatur ipsa y usque quo ad curvam *AG* in puncto *G* pertingat: qui valor extremus loco y substitutus efficit $dx \int dy \sqrt{P^2 + Q + 1} =$ Elemento Superficiei areolæ *PGgp* imminenti; quod integratum Superficiem præbet areæ integræ *APG* respondentem. Valorem autem ipsius $y = PG$ æquatio Solidi exhibet, posito: $z = o$.

Hujus vero quæstionis tractatio ad priorem illam §. II. traditam formulæ $\iint Z dx dy$ investigandæ rationem omnino accedit. Etenim priori peracta integratione ipsius $dy \sqrt{P^2 + Q^2 + 1}$, in qua x habita est constans, statuatur $y = PD = EF = e$, atque repetita integratio dabit $\int dx \int dy \sqrt{P^2 + Q^2 + 1} =$ Superficiei rectangulum *FP* tegenti, integrali ita determinato, ut posito: $x = AE$, evanescat. Perinde quidem esse, utra ipsarum x & y prior variabilis æstimetur, per se patet: ast, cum utraque via haud pari sæpe integrandi difficultate obsepta sit, multum interest, eam eligere, qua calculus simplicior redditur atque commodior.

Exem-

Exempl. 1. Si Curva AFG exprimat arcum Circuli Sphæram generantis, erit, ſumta origine abſcisſarum in centro C, atque exiſtente radio $AC = a$, æquatio hæc: $x^2 + y^2 + z^2 = a^2$, quæ dat: $z = \sqrt{a^2 - x^2 - y^2}$, $dz = \frac{-(xdx + ydy)}{\sqrt{(a^2 - x^2 - y^2)}} = Pdx + Qdy$; unde eruitur $P = \frac{-x}{\sqrt{(a^2 - x^2 - y^2)}}$, $Q = \frac{-y}{\sqrt{(a^2 - x^2 - y^2)}}$ atque $dxdy\sqrt{P^2 + Q^2 + 1} = \frac{adxdy}{\sqrt{(a^2 - x^2 - y^2)}}$. Quo integrale pateat $\int \frac{ady}{\sqrt{(a^2 - x^2 - y^2)}}$, conſtituta x conſtanti, ponatur $Sin.\ tot. = 1$, $Sin.\ v = \frac{y}{\sqrt{(a^2 - x^2)}}$, atque diameter Circuli: peripheriam : : $2 : \pi$. Quum ſit $dv = \frac{dy}{\sqrt{(a^2 - x^2 - y^2)}}$, exiſtit $\int \frac{ady}{\sqrt{(a^2 - x^2 - y^2)}} = \int adv = a.\ Arc.\ Sin.\ \frac{y}{\sqrt{(a^2 - x^2)}}$ ($=$ producto ex a in *Arcum*, cujus *Sinus* $= \frac{y}{\sqrt{(a^2 - x^2)}}$), quippe quod integrale, ductum in dx, Superficiem exhibet elementarem areolæ $PQqp$ imminentem.

Ad inveniendam primum Superficiem quadranti baſeos Circularis reſpondentem, extendatur y usque ad peripheriam, quo fiat $y^2 = a^2 - x^2$, quo valore ſubſtituto, obtinetur $\int a\,dx\ Arc.\ Sin.\ \frac{y}{\sqrt{(a^2-x^2)}}$ $=$ $\int a\,dx\ Arc.\ Sin.\ 1 = \frac{1}{4} a\pi x$; quod, ſumto: $x = a$, abit in $\frac{a^2\pi}{4}$ = octanti Superficiei Sphæricæ.

Si vero quæratur portio Superficiei rectangulo FC imminens, ſtatuatur y conſtans $= BC = EF = e$, quo facto erit $\int a\,dx\ Arc.\ Sin.\ \frac{y}{\sqrt{(a^2-x^2)}} = \int a\,dx\ Arc.\ Sin.\ \frac{e}{\sqrt{(a^2-x^2)}}$. Novimus autem eſſe $\int a\,dx\ Arc.\ Sin.\ \frac{e}{\sqrt{(a^2-x^2)}} = ax\ Arc.\ Sin.\ \frac{e}{\sqrt{(a^2-x^2)}} - \int \frac{aex^2\,dx}{(a^2-x^2)\sqrt{(a^2-e^2-x^2)}}$, ob $\int \frac{ex\,dx}{(a^2-x^2).\sqrt{(a^2-e^2-x^2)}} = Arc.\ Sin.\ \frac{e}{\sqrt{(a^2-x^2)}}$. Cum præterea ſit:

$$\frac{-aex^2\,dx}{(a^2-x^2).\sqrt{(a^2-e^2-x^2)}} = \frac{aex\,dx}{(a+x).\sqrt{(a^2-e^2-x^2)}} - \frac{a^2ex\,dx}{(a^2-x^2).\sqrt{(a^2-e^2-x^2)}},\ \frac{aex\,dx}{(a+x).\sqrt{(a^2-e^2-x^2)}} = \frac{ae\,dx}{\sqrt{(a^2-e^2-x^2)}} - \frac{a^2e\,dx}{(a+x).\sqrt{(a^2-e^2-x^2)}}$$

; erit

$$\int \frac{-aex^2\,dx}{(a^2-x^2).\sqrt{(a^2-e^2-x^2)}} = \int \frac{-a^2ex\,dx}{(a^2-x^2).\sqrt{(a^2-e^2-x^2)}} + \int \frac{ae\,dx}{\sqrt{(a^2-e^2-x^2)}} + \int \frac{-a^2e\,dx}{(a+x).\sqrt{(a^2-e^2-x^2)}}$$

—

$$-a^2 \, Arc.Sin. \frac{e}{\sqrt{(a^2-x^2)}} + ae \, Arc.Sin. \frac{x}{\sqrt{(a^2-e^2)}}$$
$$-\int \frac{a^2 e dx}{(a+e).\sqrt{(a^2-e^2-x^2)}}.$$

Ad inveniendum postremum integralis membrum ponatur: $\frac{\sqrt{(a^2-e^2)}+x}{\sqrt{(a^2-e^2)}-x} = z^2$, unde eruitur $x = \frac{\sqrt{(a^2-e^2)}(z^2-1)}{1+z^2}$, $dx = \frac{4\sqrt{(a^2-e^2)}zdz}{(1+z^2)^2}$, $a+x = \frac{a-(a^2-e^2)^{\frac{1}{2}}+(a+\overline{a^2-e^2}^{\frac{1}{2}})z^2}{1+z^2}$, $\sqrt{a^2-e^2-x^2} = \frac{2\sqrt{(a^2-e^2)}z}{1+z^2}$; atque facta substitutione emergit

$$\frac{a^2 e dx}{(a+x).\sqrt{(a^2-e^2-x^2)}} = \frac{2a^2 e dz}{a-(a^2-e^2)^{\frac{1}{2}}+(a+\overline{a^2-e^2}^{\frac{1}{2}})z^2}.$$

At constat esse: $\int \frac{2a^2 e dz}{a-\sqrt{(a^2-e^2)}+(a+\sqrt{(a^2-e^2)})z^2} = 2a^2 \, Arc.\, Tang. \left(\frac{a+\sqrt{(a^2-e^2)}}{a-\sqrt{(a^2-e)}}\right)^{\frac{1}{2}} z$ (ob $\int \frac{dz}{1+z^2}$ = *Arcui*, cujus *Tangens* = z) = $2a^2 \, Arc.\, Tang. \sqrt{\frac{(a+\sqrt{(a^2-e)})(\sqrt{(a^2-e)}+x)}{(a-\sqrt{(a^2-e^2)})(\sqrt{(a^2-e^2)}-x)}}$, restituto valore ipsius $z = \left(\frac{\sqrt{(a^2-e^2)}+x}{\sqrt{(a^2-e^2)}-x}\right)^{\frac{1}{2}}$. Unde colligitur $\int adx \, Arc.Sin. \frac{e}{\sqrt{(a^2-e^2)}} = ax \, Arc.Sin. \frac{e}{\sqrt{(a^2-x^2)}}$

—

$- a^2$ *Arc. Sin.* $\frac{e}{\sqrt{(a^2 - x^2)}} + ae$ *Arc. Sin.* $\frac{x}{\sqrt{(a^2 - e^2)}} -$

$2a^2$ *Arc. Tang.* $\sqrt{\frac{(a + \overline{a^2 - e^2}^{\frac{1}{2}})(\overline{a^2 - e^2}^{\frac{1}{2}} + x)}{(a - (a^2 - e^2)^{\frac{1}{2}})(\overline{a^2 - e^2}^{\frac{1}{2}} - x)}} +$

a^2 *Arc. Sin.* $e : a + 2a^2$ *Arc. Tang.* $\left(\frac{a + \sqrt{(a^2 - e^2)}}{a - \sqrt{(a^2 - e^2)}}\right)^{\frac{1}{2}}$, integrali ita correcto, ut posito: $x = o$, ipsum quoque evanescat. Ponatur $x = CE = \sqrt{a^2 - e^2}$, quo prodeat Superficies rectangulo FC respondens, quae igitur erit $= a\pi : 4 (\overline{a^2 - e^2}^{\frac{1}{2}} - 3a + e) + a^2$ *Arc. Sin.* $e : a$ $+ 2a^2$ *Arc. Tang.* $\left(\frac{a + \sqrt{(a^2 - e^2)}}{a - \sqrt{(a^2 - e^2)}}\right)^{\frac{1}{2}}$.

Coroll. Posito: $PG : PQ :: a : b$, formula $\int adx$ *Arc. Sin.* $\frac{y}{\sqrt{(a^2 - x^2)}}$ abit in hanc: $\int adx$ *Arc. Sin.* $\frac{b}{a}$, ob $y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$. Unde integrando obtinetur portio Superficiei Sphaericae, quae areae Ellipseos, cujus semiaxes sunt: a & b, imminet $= ax$ *Arc. Sin.* $\frac{b}{a}$.

Schol. Si initium abscissarum sumtum fuerit in vertice A, Superficiei Sphaericae investigationem, integrando formulam: $\frac{adxdy}{\sqrt{(2ax - x^2 - y^2)}}$, peragere licebit.

Exem-

Exempl. 2. Conoidis, quod Parabolam *AFG* rotando circa axem *AP* genitur, natura hac exprimitur æquatione: $y^2 + z^2 = 2ax$. Unde cum ſit $z = \sqrt{2ax - y^2}$, erit $dz = \frac{adx - ydy}{\sqrt{(2ax - y^2)}}$; quæ æquatio cum hac: $dz = Pdx + Qdy$, comparata præbet: $P = \frac{a}{\sqrt{(2ax - y^2)}}$, $Q = \frac{-y}{\sqrt{(2ax - y^2)}}$, & $dxdy\sqrt{P^2 + Q^2 + 1} = \frac{(2ax + a^2)^{\frac{1}{2}}dxdy}{\sqrt{(2ax - y^2)}}$.

Quo habeatur Superficies areæ *APG* reſpondens, integrali: $\int \frac{\overline{2ax + a^2}^{\frac{1}{2}}dy}{\sqrt{(2ax - y^2)}} = (2ax + a^2)^{\frac{1}{2}} Arc. Sin. \frac{y}{\sqrt{(2ax)}}$ eruto, ipſi y tribuatur valor $PG = \sqrt{2ax}$; quo facto altera integratio exhibebit: $\int (2ax + a^2)^{\frac{1}{2}} dx\ Arc.\ Sin. \frac{y}{\sqrt{(2ax)}} = \int \overline{2ax + a^2}^{\frac{1}{2}} dx Sin. = 1. \frac{\pi(2ax + a^2)^{\frac{3}{2}}}{12a} \pm C = \frac{\pi(2ax + a^2)^{\frac{3}{2}}}{12a} - \frac{\pi a^2}{12}$; quippe quod integrale, ex natura quæſtionis, evaneſcente x, nihilo æquatur.

Quod ſi inveſtigetur Superficies rectangulo *FP* imminens, conſtituta tum ipſa y conſtanti $= e$, formula denuo integranda: $(2ax + a^2)^{\frac{1}{2}}dx\ Arc.\ Sin. \frac{y}{\sqrt{(2ax)}}$ trans-

transformabitur in hanc: $(2ax + a^2)^{\frac{1}{2}} dx \, Arc. Sin. \frac{e}{\sqrt{(2ax)}}$.

Cum sit $\int \frac{-edx}{2x\sqrt{(2ax - e^2)}} = Arc.$, cujus $Sinus = \frac{e}{\sqrt{(2ax)}}$; obtinetur $\int (2ax + a^2)^{\frac{1}{2}} dx \, Arc. Sin. \frac{e}{\sqrt{(2ax)}} = \frac{(2ax + a^2)^{\frac{3}{2}}}{3a} Arc. Sin. \frac{e}{\sqrt{(2ax)}} + \int \frac{(2ax + a^2)^{\frac{3}{2}} edx}{6ax \sqrt{(2ax - e^2)}}$. Cujus integralis membrum posterius ut innotescat, juvat posuisse: $\frac{2ax - e^2}{2ax + a^2} = z^2$; quo videlicet facto erit $x = \frac{e^2 + a^2 z^2}{2a(1 - z^2)}$, $dx = \frac{(a^2 + e^2) z dz}{a(1 - z^2)^2}$, $(2ax + a^2)^{\frac{3}{2}} = \frac{(a^2 + e^2)^{\frac{3}{2}}}{(1 - z^2)^{\frac{3}{2}}}$, $\overline{2ax - e^2}^{\frac{1}{2}} = \frac{(a^2 + e^2)^{\frac{1}{2}} z}{(1 - z^2)^{\frac{1}{2}}}$: quibus substitutis valoribus, eruitur: $\frac{(2ax + a^2)^{\frac{1}{2}} edx}{6ax \sqrt{(2ax - e^2)}} = \frac{(a^2 + e^2)^2 edz}{3a(1 - z^2)^2 (e^2 + a^2 z^2)} = \frac{(a^2 + e^2)^2 edz}{3a(1 - z)^2 (1 + z)^2 (e^2 + a^2 z^2)}$. Fingamus jam fractionem $\frac{dz}{(1 - z)^2 (1 + z)^2 (e^2 + a^2 z^2)}$ = Summæ fractionum partialium: $\frac{Adz}{(1 - z)^2} + \frac{Bdz}{1 - z} + \frac{Cdz}{(1 + z)^2} + \frac{Ddz}{1 + z} + \frac{Edz}{e^2 + a^2 z^2}$, ex quibus, ad eundem denominatorem reductis

ductis

ductis, illa componitur; unde calculo inventis: $A = C = \frac{1}{4(a^2+e^2)}$, $B = D = \frac{3a^2+e^2}{4(a^2+e^2)^2}$ atque $E = \frac{a^4}{(a^2+e^2)^2}$, cum sit $\int\frac{Adz}{(1-z)^2} = \frac{A}{1-z}$, $\int\frac{Bdz}{1-z} = B.L\frac{1}{1-z}$, $\int\frac{Cdz}{(1+z)^2} = \frac{-C}{1+z}$, $\int\frac{Ddz}{1+z} = D.L\,1+z$, denique $\int\frac{Edz}{e^2+a^2z^2} = \frac{E}{ae}\,Arc.\,Tang.\,\frac{az}{e}$; terminis bene reductis, existit: $\int\frac{(a^2+e^2)^2edz}{3a(1-z^2)^2(e^2+a^2z^2)} = \frac{(a^2+e^2)ez}{6a(1-z^2)} + \frac{(3a^2+e^2)e}{12a}L\frac{1+z}{1-z} + \frac{a^2}{3}\,Arc.\,Tang.\,\frac{az}{e}$, atque, restituto valore ipsius $z = \left(\frac{2ax-e^2}{2ax+a^2}\right)^{\frac{1}{2}}$, $\int\frac{(2ax+a^2)^{\frac{3}{2}}edx}{6ax\sqrt{(2ax-e^2)}} =$

$$\frac{e}{6a}\sqrt{(2ax-e^2).(2ax+a^2)} +$$

$$\frac{(3a^2+e^2)e}{12a}L\frac{(2ax+a^2)^{\frac{1}{2}}+(2ax-e^2)^{\frac{1}{2}}}{(2ax+a^2)^{\frac{1}{2}}-(2ax-e^2)^{\frac{1}{2}}} +$$

$\frac{a^2}{3}\,Arc.\,Tg.\,\frac{a}{e}\sqrt{\frac{2ax-e^2}{2ax+a^2}}$. Hinc demum efficitur esse:

$$\int(2ax+a^2)^{\frac{1}{2}}dx.\,Arc.\,Sin.\,\frac{e}{\sqrt{(2ax)}} = \frac{(2ax+a^2)^{\frac{3}{2}}}{3a}\,Arc.\,Sin.\,\frac{e}{\sqrt{(2ax)}}$$

$$+\frac{e}{6a}\sqrt{(2ax-e^2).(2ax+a^2)}$$

 +

$+ \frac{(3a^2+e^2)e}{12a} L \frac{\overline{2ax+a^2}^{\frac{1}{2}}+\overline{2ax-e^2}^{\frac{1}{2}}}{(2ax+a^2)^{\frac{1}{2}}-(2ax-e^2)^{\frac{1}{2}}} +$

$\frac{a^2}{3} Arc. Tang. \frac{a}{e}\sqrt{\frac{2ax-e^2}{2ax+a^2}} + C.$ Integrale vero ita corrigendum est, ut, existente $x = AE = \frac{e^2}{2a}$, evanescat: unde $C = \frac{-(a^2+e^2)^{\frac{3}{2}}.\pi}{12a}$.

Exempl. 3. Exprimente recta AfG sectionem Coni *recti* per axem AP factam, & sumta abscissa $AP = x$, est $y^2 + z^2 = a^2 x^2$, quae aequatio dat $z = \sqrt{a^2x^2 - y^2}$, $dz = \frac{a^2xdx - ydy}{\sqrt{(a^2x^2 - y^2)}}$, atque $dxdy\sqrt{P^2+Q^2+1}$ $= \frac{a(a^2+1)^{\frac{1}{2}}xdxdy}{\sqrt{(a^2x^2-y^2)}}$.

Priori integratione eruatur $\int \frac{a(a^2+1)^{\frac{1}{2}}xdy}{\sqrt{(a^2x^2-y^2)}} =$ $a.\sqrt{(a^2+1)}\,x\,Arc.Sin. \frac{y}{ax}$, quod integrale, constituta $y = ax$, ut prodeat quadrans Superficiei Conicae, ductum in dx denuo integretur; quo pacto obtinetur: $\int a(a^2+1)^{\frac{1}{2}}xdx\ Arc.\ Sin. \frac{y}{ax} = \frac{a(a^2+1)^{\frac{1}{2}}\pi x^2}{8} + C.$ Erit vero $C = o$, ob integrale, posito $x = o$, evanescens.

Quod

Quod si in integratione altera ipsi y valor constans $= fe = PD = e$ tribuatur, invenietur Superficies rectangulo fP imminens $=$

$$\int a(a^2+1)^{\frac{1}{2}} x\,dx \text{ Arc. Sin. } \frac{e}{ax} = \tfrac{1}{2} a(a^2+1)^{\frac{1}{2}} x^2 \text{ Arc. Sin. } \frac{e}{ax}$$

$$+ \int \frac{\frac{a}{2}(a^2+1)^{\frac{1}{2}} e x\,dx}{\sqrt{(a^2 x^2 - e^2)}} \left(\text{ob} \int \frac{-e\,dx}{x\sqrt{(a^2x^2-e^2)}} = \text{Arc. Sin. } \frac{e}{ax}\right)$$

$$= \tfrac{1}{2} a(a^2+1)^{\frac{1}{2}} x^2 \text{ Arc. Sin. } \frac{e}{ax} - \frac{(a^2+1)^{\frac{1}{2}} e}{2a} \sqrt{a^2 x^2 - e^2} + C.$$

Integrale vero ita corrigatur, ut posito $x = e : a$, ipsum quoque evanescat. Quare $C = -\frac{e^2 \pi (a^2+1)^{\frac{1}{2}}}{8a}$.

Exempl. 4. Pro Cono *scaleno*, quem a *recto* eo tantum differre consideramus, quod hic Circulum, ille vero Ellipsin habeat basin, hæc obtinetur, sumtis in axe AP abscissis, æquatio: $a^2 b^2 x^2 = a^2 z^2 + b^2 y^2$, ex qua ratione usquehuc exposita elicitur valor formulæ $dx\,dy \sqrt{P^2+Q^2+1} =$

$$\frac{dx\,dy}{a} \sqrt{\frac{a^2(b^2+1)x^2 - (a^2-b^2)y^2}{a^2 x^2 - y^2}}.$$

Integrationem formulæ $\int \frac{dy}{a} \sqrt{\frac{a^2(b^2+1)x^2 - (a^2-b^2)y^2}{a^2 x^2 - y^2}}$ ex rectificatione Ellipseos pendere facile quidem patet;

tet; quae cum, nisi per seriem infinitam, exprimi nequeat, eam sequenti modo exhibebimus:

Ponatur brevitatis caussa $a^2.(b^2+1) = m^2$, $m^2 - (a^2 - b^2) = b^2.(a^2+1) = n^2$. Quo facto existit

$$\frac{dy}{a}\sqrt{\frac{a^4.(b^2+1)x^2-(a^2-b^2)y^2}{a^2x^2-y^2}} = \frac{dy}{a}\sqrt{\frac{a^2m^2x^2-(a^2-b^2)y^2}{a^2x^2-y^2}}$$

$$= \frac{dy}{a}\sqrt{m^2+\frac{n^2y^2}{a^2x^2-y^2}} = \frac{dy}{a}\sqrt{m^2+\frac{n^2y^2}{a^2x^2}+\frac{n^2y^4}{a^4x^4}+\frac{n^2y^6}{a^6x^6}+\dots}$$

$$= \frac{dy}{a}\left(m + Ay^2 + By^4 + Cy^6 + Dy^8 + Ey^{10} + \dots\right).$$

Erutis valoribus coefficientium: A, B, C, D, E, etc. ex datis hisce aequationibus: $2mA - \frac{n^2}{a^2x^2} = 0$, $2mB + A^2 - \frac{n^2}{a^4x^4} = 0$, $2mC + 2AB - \frac{n^2}{a^6x^6} = 0$, $2mD + 2AC + B^2 - \frac{n^2}{a^8x^8} = 0$, $2mE + 2AD + BC - \frac{n^2}{a^{10}x^{10}} = 0$, etc., habetur integrale seriei a nobis explicatae $= \frac{y}{a}\left(m + \frac{1}{3}.\frac{n^2y^2}{2ma^2x^2} + \frac{1}{5}.\frac{\overline{4m^2-n^2}.n^2.y^4}{8m^3a^4x^4}\right.$ $\left. + \frac{1}{7}.\frac{\overline{8m^4-(4m^2-n^2)n^2}.n^2}{16m^5a^6x^6}y^6 + \dots\right)$.

Extendatur jam PQ ad G, atque erit $y = ax$: tum hic substituatur valor in serie inventa et ducatur haec ipsa in dx; quo peracto, repetita integratio

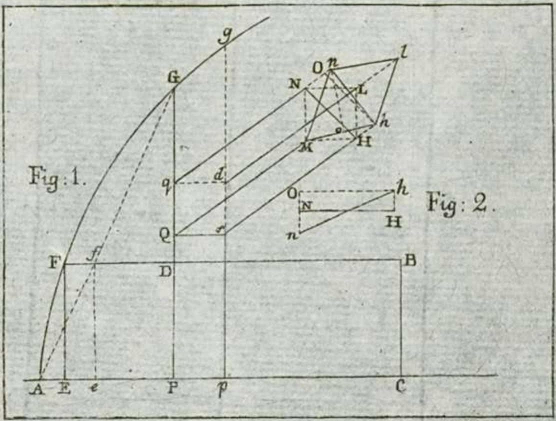
tio dabit Superficiei Conicæ quadrantem $= \int x dx \,(m +$
$\frac{1}{3} \cdot \frac{n^2}{2m} + \frac{1}{5} \cdot \frac{\overline{4m^2 - n^2} \,.\, n^2}{8m^3} + \frac{1}{7} \cdot \frac{\overline{8m^4 - (4m^2 - n^2) n^2} \,.\, n^2}{16m^5} \ldots)$

$$= \frac{x^2}{2} (m + \frac{1}{3} \cdot \frac{n^2}{2m} + \frac{1}{5} \cdot \frac{\overline{4m^2 - n^2} \,.\, n^2}{8m^3} + \frac{1}{7} \cdot \frac{\overline{8m^4 - (4m^2 - n^2) n^2} \,.\, n^2}{16m^5} \ldots)$$

Dato autem ipsi y, uti in prioribus exemplis, valore constante $= e$, prodibit Superficies basi rectangulari, quæ producto *ex* exprimitur, imminens $= \int \frac{edx}{a} (m +$
$\frac{1}{3} \cdot \frac{n^2 e^2}{2ma^2x^2} + \frac{1}{5} \cdot \frac{\overline{4m^2 - n^2} \,.\, n^2 e^4}{8m^3 a^4 x^4} + \ldots\ldots\ldots\ldots)$
$= \frac{ex}{a} (m - \frac{1}{3} \cdot \frac{n^2 e^2}{2ma^2x^2} - \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{\overline{4m^2 - n^2} \,.\, n^2 e^4}{8m^3 a^4 x^4} -$
$- \frac{1}{7} \cdot \frac{1}{5} \cdot \frac{\overline{8m^4 - (4m^2 - n^2) n^2} \,.\, n^2 e^6}{16m^5 a^6 x^6} + \ldots\ldots\ldots) \pm C.$

Quum autem ex natura quæstionis pateat, integrale hoc, posito $x = e : a$, nihilo æquari, id ipsum corrigendum est subducendo $C = \frac{e^2}{a^2} (m - \frac{1}{3} \cdot \frac{n^2}{2m} -$
$\frac{1}{5} \cdot \frac{1}{3} \cdot \frac{\overline{4m^2 - n^2} \,.\, n^2}{8m^3} - \frac{1}{7} \cdot \frac{1}{5} \cdot \frac{\overline{8m^4 - (4m^2 - n^2) n^2} \,.\, n^2}{16m^5} + \ldots).$

Schol. Si ponatur $a = b$, erit $m = n$, atque Cono *scaleno* abeunte in *rectum*, formulas nostras, simpliciore jam exhibendas forma cum iis, quas pro hoc proxime supra elicuimus, quo consensus detegatur,



CLS fc.

tur, comparandi via patebit. Sic prior formula, facta substitutione, transformata hæc existit: $a\sqrt{(a^2+1)}\,x^2\left(\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{3}{16}+\ldots\right)$; cujus cum hac $\frac{a\sqrt{(a^2+1)}\,\pi x^2}{8}$ consensus facile perspicitur, cum, iisdem utrinque demtis factoribus, tantummodo restet, ut probetur esse $\pi:8=\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{3}{16}+\frac{1}{7}\cdot\frac{5}{32}+\ldots$; quippe quæ series, vel paucis tantum terminis collectis, diametri Circuli ad peripheriam rationem, a Geometris variis artificiis exploratam, exacte satis exprimit, adeo ut æquationis nostræ veritas sponte pateat: atque sic quasi forte quadam hoc exemplo effectum est, ut seriei propositæ emineat congruentia cum altera hac, primo quidem intuitu non parum diversa: $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots\ldots$; cum utraque eandem summam, octavam videlicet peripheriæ Circuli partem exhauriat.

EMENDANDA.

Pag. 6. L. 3. pro: $\frac{ax^{\frac{\overline{m+1}.r+\overline{n+1}.s}{r}}}{(m+1).(n+1)}$ leg. $\frac{arx^{\frac{\overline{m+1}.r+\overline{n+1}.s}{r}}}{(m+1).(\overline{m+1}.r+\overline{n+1}.s)}$

\- - - 6. pro: $\frac{ay^{\frac{\overline{m+1}.r+\overline{n+1}.s}{s}}}{(m+1).(n+1)}$ leg. $\frac{asy^{\frac{\overline{m+1}.r+\overline{n+1}.s}{s}}}{(n+1).(\overline{m+1}.r+\overline{n+1}.s)}$

\- 13. - 2. pro: $\int\frac{a^2edx}{(a+e)\sqrt{(a^2\cdot e^2\cdot x^2)}}$ l. $\int\frac{a^2edx}{(a+x).\sqrt{(a^2-e^2\cdot x^2)}}$

\- - - ult. pro: $\int adx\,A.S.\frac{e}{\sqrt{(a^2-e^2)}}$ l. $\int adx\,A.S.\frac{e}{\sqrt{(a^2\cdot x^2)}}$

\- 14. - 7. pro: $a\pi:4(\overline{a^2-e^2}^{\frac{1}{2}}-3a+e)$ l. $\frac{1}{4}a\pi(\overline{a^2-e^2}^{\frac{1}{2}}-3a+e)$

\- 15. - 12. pro: $\int\overline{2ax+a^2}^{\frac{1}{2}}dx\,S.=1.$ l. $\int\overline{2ax+a^2}^{\frac{1}{2}}dx\,A.S.1=$