DISSERTATIO MATHEMATICA

ANALYSEOS SUBLIMIORIS ALGEBRÆ ELEMENTARI CONNECTENDÆ SPECIMEN

EXHIBENS.

CUJUS PARTICULAM IV.

CONS. AMPL. FACULT. PHILOS. ABOËNS.

P. P.

GABRIEL PALANDER,

Adj. Fac. Philos. Ord.

ET

JOHANNES JAC. TENGSTRÖM,

Stipend. Reg. Ostrobothniensis.

In Audit. Phys. die XV Junii MDCCCVII.

h. a. m. s.

ABOÆ,

Typis FRENCKELLIANIS.

6

functionem: $a_1^2 a_2^0 a_3^0 + a_1' a_2' a_3^0 + a_1' a_2^0 a_3' + a_1^0 a_2^2 a_3^0 + a_1^0 a_2' a_3' + a_1^0 a_2^0 a_3^2 = a_1^2 + a_1 a_2 + a_1 a_3 + a_2^2 + a_2 a_3 + a_3^2$.

42. Quod si fiat $s_1 + s_2 \,..+ s_m = n'$; erit $s_{m+1} + s_{m+2} \,..+ s_r = n - n'$ (denotante m numerum integrum positivum $< r$). Quo pacto, cum quaelibet functio formae $a_1^{s_1} a_2^{s_2} \,.. a_m^{s_m}$ non possit non contineri sub formula $(a_1, a_2 \,.. a_m)^{n'}$ pariterque quaelibet formae $a_{m+1}^{s_{m+1}} . a_{m+2}^{s_{m+2}} \,.. a_r$ sub formula $(a_{m+1}, a_{m+2}, .. a_r)^{n-n'}$; aperte constat, istos terminos functionis $(a_1, a_2 \,.. a_r)^n$, in quibus summa indicum ab s_1 ad s_m inclusive exsurgit ad n', comprehendi sub formula $(a_1, a_2, .. a_m)^{n'} . (a_{m+1}, a_{m+2}, .. a_r)^{n-n'}$, quippe quam universi exhauriunt. Unde facile conficitur, ponendo nimirum pro n' numeros $n, n-1, \ldots 2, 1, 0$ successive, esse $(a_1, a_2, .. a_r)^n = (a_1, a_2, .. a_m)^n . (a_{m+1}, a_{m+2}, .. a_r)^0 + (a_1, a_2 \,.. a_m)^{n-1} . (a_{m+1}, a_{m+2}, .. a_r)' + .. + (a_1, a_2, .. a_m)^2 . (a_{m+1}, a_{m+2}, .. a_r)^{n-2} + (a_1, a_2, .. a_m)' . (a_{m+1}, a_{m+2}, .. a_r)^{n-1} + (a_1, a_2, .. a_m)^0 . (a_{m+1}, a_{m+2}, .. a_r)^n$.

43. Sit e. gr. $m = 1$. Quo posito habebitur $(a_1, a_2, .. a_r)^n = a_1^n (a_2, a_3, .. a_r)^0 + .. + a_1 (a_2, a_3, .. a_r)^{n-1} + (a_2, a_3, .. a_r)^n$.

 44. Sit

44. Si vero in Ex. præced. ſimul ponatur $r = 2$; emergit $(a_1, a_2)^n = a_1^n + a_1^{n-1} a_2 + \ldots + a_1 . a_2^{n-1} + a_2^n = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2}$.

45. Cum ſit $a_1^{n}(a_2, a_3, \ldots a_r)^0 + a_1^{n-1}(a_2, a_3, \ldots a_r)' + \ldots + a_1 . (a_2, a_3, \ldots a_r)^{n-1} = a_1 [(a_1^{n-1} (a_2, a_3, \ldots a_r)^0 + a_1^{n-2}(a_2, a_3, \ldots a_r)' + \ldots + (a_2, a_3, \ldots a_r)^{n-1}] = a_1 (a_1, a_2, \ldots a_r)^{n-1}$ (n:o 43); patet hinc fore $(a_1, a_2, \ldots a_r)^n = a_1 (a_1, a_2, \ldots a_r)^{n-1} + (a_2, a_3, \ldots a_r)^n = a_1 (a_2, \ldots a_r)^{n-1} + a_2 (a_2, a_3, \ldots a_r)^{n-1} + (a_3, a_4, \ldots a_r)^n$ vel generaliter $= a_1 (a_1, a_2, \ldots a_r)^{n-1} + a_2 (a_2, a_3, \ldots a_r)^{n-1} + \ldots + a_m (a_m, a_{m+1}, \ldots a_r)^{n-1} + (a_{m+1}, a_{m+2}, \ldots a_r)^n$, quippe quæ formula poſito $m = r-1$ tranſit in hanc: $a_1 (a_1, a_2, \ldots a_r)^{n-1} + a_2 (a_2, a_3 \ldots a_r) + \ldots + a_{r-1}(a_{r-1}, a_r)^{n-1} + a_r^n$.

46. Deſignet jam brevitatis gratia ${}^nC_r$ valorem numericum, quem obtinet $(a_1, a_2, \ldots a_r)^n$ poſito $a_1 = a_2 = \ldots a_r = 1$, nec non $S. {}^nC_r$ ſummam hujus ſeriei: ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_r$. Quibus ſubſtitutis valoribus in æquatione n:o præced. exhibita $(a_1, a_2, \ldots a_r)^n = a_1 (a_1, a_2, \ldots a_r)^{n-1} + a_2 (a_2, a_3, \ldots a_r)^{n-1} + \ldots + a_{r-1} (a_{r-1}, a_r)^{n-1} + (a_r)^n$, evincitur eſſe ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1} + \ldots + {}^{n-1}C_2 +$

${}^{n-1}C_1$

${}^{n-1}C_1 = S.\ {}^{n-1}C_r$. Cum vero generatim sit $(a_1, a_2, .. a_r)^0 = a_1^0 . a_2^0 .. a_r^0 = 1$; erit etiam, pro quolibet valore numeri r, ${}^0C_r = 1$. Unde ${}^1C_r = {}^0C_1 + {}^0C_2 + .. {}^0C_r = r, = S.\ {}^0C_r$. Hinc jam eruitur ${}^2C_r$

$$= 1 + 2 + .. + r = \frac{r . \overline{r+1}}{1 . 2} = S.\ {}^1C_r.$$ Ex data autem æquatione $${}^mC_r = \frac{r . \overline{r+1} .. \overline{r+m-1}}{1 . 2 .. m}$$

facillime derivatur $${}^{m+1}C_r = \frac{r\ (r+1) .. \overline{r+m}.}{1 . 2 .. (m+1)}.$$

Sit nimirum ${}^{m+1}C_r = {}^mC_1 + {}^mC_2 + .. + {}^mC_{r-1} +$

$$\frac{r . \overline{r+1} .. \overline{r+m-1}}{1 . 2 .. m} = f(r) + \frac{r . \overline{r+1} ... \overline{r+m}}{1 . 2 ... (m+1)},$$

ideoque ${}^{m+1}C_{r-1} = {}^mC_1 + {}^mC_2 + .. {}^mC_{r-1} = f(r-1) +$

$$\frac{\overline{r-1} . r .. \overline{r+m-1}}{1 . 2 .. (m-1)},$$ exprimente $f\ (r-1)$ valorem, quem adsequitur $f\ (r)$ posito $r-1$ pro r. Quibus sic constitutis erit ${}^{m+1}C_r - {}^{m+1}C_{r-1} =$

$$\frac{r . \overline{r+1} .. \overline{r+m-1}}{1 . 2 ... m} = f\ (r) - f\ (r-1) +$$

$$\frac{r . \overline{r+1} .. \overline{r+m}}{1 . 2 .. (m+1)} - \frac{\overline{r-1} . r .. \overline{r+1-m}}{1 . 2 ... (m+1)} = f\ (r)$$

 —f

$$= f(r-1) + \frac{r.\overline{r+1}..\overline{r+m-1}\,[(r+m)-(r-1)]}{1.2..(m+1)}$$

$$= f(r) - f(r-1) + \frac{r.\overline{r+1}..\overline{r+m-1}}{1.2..m}.$$ Unde liquet esse $f(r) - f(r-1) = 0$ sive $f(r)=f(r-1)$ quantitatem constantem. Est igitur ${}^{m+1}C_r =$

$$\frac{r.\overline{r+1}..\overline{r+m}}{1.2..(m+1)} + C,$$ in qua formula esse $C=0$ exinde patet, quod functio ${}^{m+1}C_r$ ejus sit naturæ ut evanescat facto $r = 0$. Quare, cum probata sit æquatio ${}^{2}C_r = \frac{r.\overline{r+1}}{1.2}$; erit ${}^{3}C_r = \frac{r.\overline{r+1}.\overline{r+2}}{1.2.3}$, denique ${}^{n}C_r = \frac{r.\overline{r+1}...\overline{r+n-1}}{1.2..n} =$

$$\frac{(n+1)(n+2)..(r+n-1)}{1.2..(r-1)}\ ^{*)} = {}^{r-1}C_{n+1}.$$

47. Quia

*) Esse $\frac{r(r+1)..(r+n-1)}{1.2..n} = \frac{(n+1)(n+2)..(r+n-1)}{1.a..(r-1)}$ hunc in modum probari potest. Sit nimirum 1:o $r = n$. Quo pacto erit $\frac{r(r+1)..(r+n-1)}{1.2..n} =$

47. Quia est (n:o 43) $(a_1, a_2, .. a_r)^n = a_1^n + a_1^{n-1} (a_2, a_3, .. a_r)' + .. + a_1 (a_2, a_3, .. a_r)^{n-1} + (a_2, a_3, .. a_r)^n$ nec non $(a_{r+1}, a_2, .. a_r)^n = a_{r+1}^n + a_{r+1}^{n-1} (a_2, a_3, .. a_r)' + .. + a_{r+1} (a_2, a_3, .. a_r)^{n-1} + (a_2, a_3, .. a_r)^n$; idcirco erit

$$\frac{(a_1, a_2, .. a_r)^n - (a_{r+1}, a_2, .. a_r)^n}{a_1 - a_{r+1}} = \frac{a_1^n - a^n_{r+1}}{a_1 - a_{r+1}} + \frac{(a_1^{n-1} - a_{r+1}^{n-1})(a_2, a_3, .. a_r)^1}{a_1 - a_{r+1}} + . . +$$

$(a_1$-

$$\frac{r.(n+1)..(r+n-1)}{1.2..(r-1).r} = \frac{(n+1)(n+2)..(r+n-1)}{1.2..(r-1)}$$

Sit 2:o $r < n$. Jam vero erit $\frac{r(r+1)..(r+n-1)}{1.2..n} =$

$$\frac{r.(r+1)..n(n+1)..(r+n-1)}{1.2..(r-1).r(r+1)..n} = \frac{n.(n+1)..(r+n-1)}{1.2..r-1}$$

Quod si denique sit 3:o $r > n$; habebitur $\frac{r.(r+1)..(r+n-1)}{1.2..n}$

$$= \frac{r(r+1)..(r+n-1)}{1.2..n} \times \frac{(n+1)(n+2)..(r-1)}{(n+1)(n+2)..(r-1)} =$$

$$\frac{(n+1)(n+2)..(r-1)r(r+1)..(r+n-1)}{1.2..n(n+1)(n+2)..(r-1)}$$

$$= \frac{(n+1)(n+2)..(r+n-1)}{1.2..(r-1)}.$$

$$\frac{(a_1 - a_{r+1})(a_2, a_3, \ldots a_r)^{n-1}}{a_1 - a_{r+1}} = (a_1, a_{r+1})^{n-1} +$$
$(a_1, a_{r+1})^{n-2}(a_2, a_3, \ldots a_r)' + \ldots + (a_2, a_3, \ldots a_r)^{n-1}$ (n:o 44) $= (a_1, a_2, \ldots a_{r+1})^{n-1}$ (n:o 42).

48. Fiat $f(x) = x^m$, existente m numero positivo integro. Qua sub conditione erit (n:o 44 & 47) $f_1(x, x_1) = (x^m - x_1^m) : (x - x_1) = (x, x_1)^{m-1}$, $f_2(x, x_1,$

$$x_2) = \frac{(x, x_1)^{m-1} - (x_1, x_2)^{m-1}}{x - x_2} = (x, x_1, x_2)^{m-2},$$

atque generatim $f_r(x, x_1, \ldots x_r) = (x, x_1, \ldots x_r)^{m-r}$. Facto vero demum $r = m$ exsurgit $f_m(x, x_1, \ldots x_m) = (x, x_1, \ldots x_r)^0 = 1$. Functiones derivatas altiorum ordinum evanescere, facile quisque, vel me non monente, perspiciet.

49. Sit $f_{r-1}(x, x_1, \ldots x_{r-1}) = f_{r-1}^{(1)}(x, x_1, \ldots x_{r-1}) + C$. Quo pacto erit

$$\frac{f_{r-1}(x, x_1, \ldots x_{r-1}) - f_{r-1}(x_r, x_1, \ldots x_{r-1})}{x - x_r} =$$

$$\frac{[f_{r-1}^{(1)}(x, x_1, \ldots x_{r-1}) + C] - [f_{r-1}^{(1)}(x_r, x_1, \ldots x_{r-1}) + C]}{x - x_r}$$

$$= f_{r-1}^{(1)}$$

$$= \frac{f^{(1)}_{r-1}(x, x_1, . x_{r-1}) - f^{(1)}_{r-1}(x_r, x_1, .. x_{r-1})}{x - x_r}, \text{ sive}$$

$f_r(x, x_1, .. x_r) = f_r^{(1)}(x, x_1, . x_r)$. Unde patet, functionem derivatam ordinis $r - 1$, etsi quantitate arbitraria constante C aucta vel mulctata fuerit, attamen ad eandem ducere functionem derivarivatam ordinis proxime insequentis r. Quare, si inventus fuerit specialis quidam valor functionis derivatae ordinis $r - 1$, qui functioni $f_r(x, x_1, .. x_r)$ satisfaciat, est illi adjungenda quantitas indefinita constans, quo generaliter expressa obtineatur functio $f_{r-1}(x_{r-1}, x_1, .. x_{r-1})$.

50. Cum vero sit (n:o 36) $f_r(x, x_1, .. x_r)$ functio symmetrica quantitatum $x, x_1, .. x_r$; pro quibusvis harum valoribus sequentes permanebunt aequationes:

$$A_1) f_{r-1}(x, x_2, .. x_r) - f_{r-1}(x_1, x_2, .. x_r) = (x - x_1) f_r(x, x_1, .. x_r),$$

$$A_2) f_{r-1}(x, x_1, x_3, .. x_r) - f_{r-1}(x, x_2, x_3, .. x_r) = (x_1 - x_2) f_r(x, x_1, .. x_r)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$A_r) f_{r-1}(x, x_1, . x_{r-2}, x_{r-1}) - f_{r-1}(x, x_1 . x_{r-2}, x_r) = (x_{r-1} - x_r) f_r(x, x_1 . x_r)$$

Fiat igitur $x_1 = x_2 = x_r = a$ in aequ. A_1; $x_2 =$

$x_3 =$

$x_3 = \,.\,.\, x_r = a$ in aequ. A_2; atque generatim $x_n = x_{n+1} = \,.\,.\, x_r = a$ in aequ. A_n. Qua ratione obtinebitur

$$f_{r-1}(x, a\,.\,.\,.) - f_{r-1}(a\,.\,.\,.) = (x-a) f_r(x, a\,.\,.\,.),$$
$$f_{r-1}(x, x_1, a\,.\,.) - f_{r-1}(x, a\,.\,.\,.) = (x_1 - a) f_r(x, x_1, a\,.\,.),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f_{r-1}(x, x_1, .\,.\, x_{r-1}) - f_{r-1}(x, x_1, .\,.\, x_{r-2}, a) = (x_{r-1} - a) f_r(x, x_1, .\,.\, x_{r-1}, a)$$

Quibus collectis aequationibus emergit $f_{r-1}(x, x_1, .\,.\, x_{r-1}) - f_{r-1}(a\,.\,.\,.) = (x-a) f_r(x, a\,.\,.\,.) + (x_1 - a) f_r(x, x_1, a\,.\,.) + \,.\,.\,.\, + (x_{r-1} - a) f_r(x, x_1, .\,.\, x_{r-1}, a)$. Unde $f_{r-1}(x, x_1, .\,.\, x_{r-1}) = f_{r-1}(a\,.\,.\,.) + (x-a) f_r(x, a\,.\,.\,.) + (x_1 - a) f_r(x, x_1, a\,.\,.) + \,.\,.\, + (x_{r-1} - a)\,.\, f_r(x, x_1, .\,.\, x_{r-1}, a)$, in qua formula $f_{r-1}(a\,.\,.\,.) = C$ (n:o praeced.). Ex data sic functione $f_{r-1}(x, x_1, .\,.\, x_{r-1})$, eadem repetita operatione eruere licebit $f_{r-2}(x, x_1, .\,.\, x_{r-2})$ nec non reliquas functiones derivatas ordinum inferiorum, & ipsam denique functionem originariam $f(x)$.

Quae hujus ope formulae patet via a functionibus derivatis altiorum ordinum ad eas inferiorum regrediendi, ea Methodum constituit functionum derivatarum *inversam*, *directae* videlicet illi oppositam, qua a functione originaria ad derivatas est progrediendum.

51. Sit

51. Sit e. gr. $f_r(x, x_1, \ldots x_r) = f_2(x, x_1, x_2) = \frac{x x_1 + x x_2 + x_1 x_2}{x^2 x_1^2 x_2^2}$. Quo pacto habebitur $f_1(x, x_1) = C + (x - a) f_2(x, a, a) + (x_1 - a) f_2(x, x_1, a) = C + \frac{(x-a)(2ax + a^2)}{a^4 x^2} + \frac{(x_1 - a)(x x_1 + a(x + x_1))}{a^2 x^2 x_1^2}$. Unde porro elicitur $f(x) = C' + (x - a) f_1(x, a) = C' + C(x - a) + \frac{(x-a)^2(2ax + a^2)}{a^4 x^2} = \frac{a(a^3 C + 2)x^3 - a^2(a^3 C - a^2 C' + 3)x^2 + a^4}{a^4 x^2}$ $= Ax + B + \frac{1}{x^2}$ (facto brevitatis gratia $\frac{a^3 C + 2}{a^3} = A$, & $\frac{a^2 C' - a^3 C - 3}{a^2} = B$), in qua formula quantitates A & B, utpote functiones quantitatum indefinitarum constantium a, C & C', sunt constantes arbitrariæ. Quod si fiat $C = \frac{-2}{a^3}$ & $C' = \frac{1}{a^2}$; erit $A = B = 0$, atque valor functionis $f(x)$ ex hac determinatione oriundus $= \frac{1}{x^2}$.

52. Per se vero patet, posse in formula (n:o 50) exhibita pro a valorem quemvis numericum adhiberi, modo ita sit comparatus, ut functiones $f_r(x, a, \ldots)$, $f_r(x, x_1, a \ldots)$ &c. non reddat infinitas.

 Sic

Sic, si in Ex. præced. posuissemus $a = 1$, fuissemus habituri $f_1 (x, x_1) = C + (x - 1) f_2 (x, 1, 1) + (x_1 - 1) f_2 (x, x_1, 1) = C + \frac{(x - 1)(2x + 1)}{x^2} + \frac{(x_1 - 1)(xx_1 + x + x_1)}{x_1^2 \; x_1^2}$ atque $f(x) = C' + (x - 1)$

$$f_1 (x, 1) = C' + C . (x - 1) + \frac{(x - 1)^2 (2x + 1)}{x^2} =$$

$(C + 2) x + (C' - C - 3) + \frac{1}{x^2} = Ax + B + \frac{1}{x^2}$ sumto $A = C + 2$ & $B = C' - C - 3$.

53. Fiat jam $a = o$ in æquatione $f_{r-1} (x, x_1, \ldots x_{r-1}) = C + (x - a) f_r (x, a \ldots) + (x_1 - a) f_r (x, x_1, a \ldots) + \ldots + (x_{r-1} - a) . f_r (x, x_1, \ldots x_{r-1}, a)$. Quo pacto erit $f_{r-1} (x, x_1, \ldots x_{r-1}) = C + x_1 f_r (x, o \ldots) + x_1 f_r (x, x_1, o \ldots) + \ldots + x_{r-1} f_r (x, x_1, \ldots x_{r-1}, o)$. Quæ formula usui maxime est accommodata, iis exceptis casibus, in quibus functiones $f_r (x, o \ldots)$, $f_r (x, x_1, o \ldots)$, &c. infinitæ evadunt. Sumto e. gr. $f_r (x, x_1, \ldots x_r) = f_3 (x, x_1, x_2, x_3) = x + x_1 + x_2 + x_3$ habebitur $f_2 (x, x_1, x_2) = C + x f_3 (x, o, o, o) + x_1 f_3 (x, x_1, o, o) + x_2 f_3 (x, x_1, x_2, o) = C + x . x + x_1 (x + x_1) + x_2 (x + x_1 + x_2)$. Unde porro elicitur $f_1 (x, x_1) = C' + x f_2 (x, o, o) + x_1 f_2 (x, x_1, o) = C' + x . (C + x^2) + x_1 (C + x^2 + x_1 (x + x_1))$. Hinc denique $f (x) = C'' + x f_1 (x, o) = C'' + x (C' + x (C + x^2)) = C'' + C' x + C x^2 + x^4$.

54. Quia